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FLOW OF VISCO-ELASTIC LIQUIDS IN TUBES

BY

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Abstract. In the present paper, we have obtained the conditions under which a rectilinear flow of a visco-elastic fluid, which is isotropic in its state of rest and for which the stress components are expressible as polynomials in the gradients of velocity, second acceleration, ...n^{th} acceleration at the point considered, can be maintained by a uniform pressure gradient. We also show that the flow of such fluids in tubes can be identified with those of certain non-Newtonian liquids. In addition we have obtained expressions for the dissipation of energy for certain simple flows and use these to derive certain restrictions on the functions of scalar invariants occurring in the expression for the stress tensor from the fact that the dissipation function should be essentially non-negative.

Introduction. The relation between stress and rate of deformation tensors for an isotropic flow in which the stress at any point is assumed to depend only on the velocity gradients at that point was considered by Reiner (1945) and Rivlin (1948). If the flow is incompressible and the stress components are expressible as polynomials in the velocity gradient only, then the stress tensor can be expressed in the form.

$$t_{ij} = \theta d_{ij} + \psi d_{ik} d_{kj} - p\delta_{ij}, \tag{1}$$

where d_{ij} is the rate-of-deformation tensor defined in terms of velocity vector by

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \; d_{ii} = 0 \tag{2}$$

p is an arbitrary hydrostatic pressure and comma denotes differentiation with respect to one of the co-ordinates. Also θ and ψ are polynomials in invariants II and III defined by

$$II = \tfrac{1}{2} d_{ji} d_{ij} \qquad III = \tfrac{1}{3} d_{ij} d_{jk} d_{ki} \tag{3}$$

More recently, Rivlin and Erickson (1955) have shown that if we assume that in a visco-elastic fluid which is isotropic in its state of rest, the stress components are expressible as polynomials in the gradients of velocity, second acceleration,............$(n-1)^{\text{th}}$ acceleration at the point considered, then the stress matrix $T = \|t_{ij}\|$ may be expressed as a matrix polynomials in n kinematic matrices A_1, A_2,.........,A_n and whose coefficients are expressible as polynomials in traces of products formed out of these matrices. The kinematic matrices are defined by the equations.

$$A_r = \|\overset{(r)}{A_{ij}}\| \qquad (r = 1,2,\ldots\ldots,n) \tag{4}$$

$$\overset{(1)}{A_{ij}} = v_{i,j} + v_{j,i} = 2d_{ij} \tag{5}$$

$$A_{ij}^{(r+1)} = \frac{\partial}{\partial t} A_{ij}^{(r)} + v_l \overset{(r)}{A_{ij,l}} + \overset{(r)}{A_{mi}} v_{m,j} + \overset{(r)}{A_{mj}} v_{m,i} \qquad [r = 1,2\ldots,n-1] \tag{6}$$

In a later paper, Rivlin (1955) considered certain simple types of steady-state laminar flows *viz.* rectilinear laminar flow, torsional flow between two parallel plane discks, helical flow in the annular region between the two coaxial cylinders, for these mere general visco-elastic fluids In all these cases, $A_r = 0$ for $r \geq 2$ and

$$T = -p^I + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_1^2 + \alpha_4 A_2^2 + \alpha_5 (A_1 A_2 + A_2 A_1)$$
$$+ \alpha_6 (A_1^2 A_2 + A_2 A_1^2) + \alpha_7 (A_1 A_2^2 + A_2^2 A_1 + \alpha_8 (A_1^2 A_2^2 + A_2^2 A_1^2) \qquad (7)$$

where I is the unit matrix and $\alpha_1, \alpha_2 \ldots\ldots\ldots, \alpha_8$ are polynomials in the ten scalar invariants tr. A_1, tr. A_1^2, tr. A_1^3, tr. A_2, tr. A_2^2, tr. A_2^3, tr. $A_1 A_2$, tr. $A_1^2 A_2$, tr. $A_1 A_2^2$ and tr. $A_1^2 A_2^2$.

It is obvious that the fluids characterised by (1) are particular cases of those characterised by (7) (*i.e.* when $\alpha_2 = 0$, $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0$).

Rivlin and Green (1956) have considered the flow of non-Newtonian fluids characterised by (1) through tubes and in particular show that in general, unless the tube is circular, a rectilinear flow cannot be maintained by a uniform pressure gradient and secondary flows can arise in the absence of body forces. This conclusion was earlier reached by Erickson (1956) and later examined by Stone (1956).

In the present paper, we show that a rectilinear flow of a liquid characterised by (7) can be maintained by a uniform pressure gradient under the same conditions as obtained by Rivlin and Green (1956) for the less general flow characterised by (1). In fact, we prove a more general result that the rectilinear flows in tubes of the visco-elastic flows characterised by (7) can be identified with those of certain non-Newtonian fluids characterised by (1),

We also obtain expression for the dissipation of energy and obtain certain restrictions on α's from the fact that this should not be negative.

2. **The Basic Equations.** Let the rectilinear flow be given by

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = f(x,y) \qquad (8)$$

so that

$$A_1 = \begin{bmatrix} 0 & 0 & f_x \\ 0 & 0 & f_y \\ f_x & f_y & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 2f_x^2 & 2f_x f_y & 0 \\ 2f_x f_y & 2f_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (9)$$

and

$$A_r = 0 \qquad [r > 2] \qquad (10)$$

From (9) we easily obtain the matrices A_1^2, A_2^2, $A_1 A_2$, $A_2 A_1$, $A_1 A_2^2$, $A_2^2 A_1$, $A_1^2 A_2$, $A_2 A_1^2$, $A_2^2 A_1^2$, $A_1^2 A_2^2$ and their traces which are given by

$$\left.\begin{aligned}
&\text{tr. } A_1 = 0,\ \text{tr. } A_2 = 2(f_x^2+f_y^2),\ \text{tr. } A_1^2 = 2(f_x^2+f_y^2),\\
&\text{tr. } A_2^2 = 4(f_x^2+f_y^2)^2,\ \text{tr. } A_1A_2^2 = 0,\ \text{tr. } A_1A_2 = 0,\\
&\text{tr. } A_2A_1 = 0,\ \text{tr. } A_1^2A_2 = 2(f_x^2+f_y^2)^2,\\
&\text{tr. } A_1^2A_2^2 = 4(f_x^2+f_y^2)^3,\ \text{tr. } A_1^3 = 0,\\
&\text{tr. } A_2^3 = 8(f_x^2+f_y^2)^3
\end{aligned}\right\} \tag{11}$$

Also the components of the stress tensor are given by

$$\left.\begin{aligned}
t_{11} &= -p+(2\alpha_2+\alpha_3)f_x^2+4(\alpha_4+\alpha_6)f_x^2(f_x^2+f_y^2)+8\alpha_8f_x^2(f_x^2+f_y^2)^2\\
t_{22} &= -p+(2\alpha_2+\alpha_3)f_y^2+4(\alpha_4+\alpha_6)f_y^2(f_x^2+f_y^2)+8\alpha_8f_y^2(f_x^2+f_y^2)^2\\
t_{33} &= -p+\alpha_3(f_x^2+f_y^2)\\
t_{12} &= (2\alpha_2+\alpha_3)f_xf_y+4(\alpha_4+\alpha_6)f_xf_y(f_x^2+f_y^2)+8\alpha_8f_xf_y(f_x^2+f_y^2)^2\\
t_{13} &= \alpha_1f_x+2\alpha_5f_x(f_x^2+f_y^2)4\alpha_7f_x(f_x^2+f_y^2)^2\\
t_{23} &= \alpha_1f_y+2\alpha_5f_y(f_x^2+f_y^2)+4\alpha_7f_y(f_x^2+f_y^2)^2
\end{aligned}\right\} \tag{12}$$

From (11) it appears that $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\ldots, \alpha_8$ are all function of $f_x^2+f_y^2$. Again using (8) and (12) the equations of motion give

$$\varrho X+\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\{(2\alpha_2+\alpha_3)f_x^2+4(\alpha_4+\alpha_6)f_x^2(f_x^2+f_y^2)+8\alpha_8f_x^2(f_x^2+f_y^2)^2\}$$

$$+\frac{\partial}{\partial y}\{(2\alpha_2+\alpha_3)f_y^2+4(\alpha_4+\alpha_6)f_xf_y(f_x^2+f_y^2)+8\alpha_8f_xf_y(f_x^2+f_y^2)^2\}$$

$$\varrho Y+\frac{\partial p}{\partial y} = \frac{\partial}{\partial x}\{(2\alpha_2+\alpha_3)f_xf_y+4(\alpha_4+\alpha_6)f_xf_y(f_x^2+f_y^2)+8\alpha_8f_yf_x(f_x^2+f_y^2)^2\} \tag{13}$$

$$+\frac{\partial}{\partial y}\{(2\alpha_2+\alpha_3)f_y^2+4(\alpha_4+\alpha_6)f_y^2(f_x^2+f_y^2)+8\alpha_8f_y^2(f_x^2+f_y^2)^2\} \tag{14}$$

$$\varrho Z+\frac{\partial p}{\partial z} = \frac{\partial}{\partial x}\{\alpha_1f_x+2\alpha_5f_x(f_x^2+f_y^2)+4\alpha_7f_x(f_x^2+f_y^2)^2\}$$

$$+\frac{\partial}{\partial y}\{\alpha_1f_y+2\alpha_5f_y(f_x^2+f_y^2)+4\alpha_7f_y(f_x^2+f_y^2)^2\} \tag{15}$$

3. Identification with the motion of non-Newtonian fluids. Let us make the substitutions

$$\alpha_1+2\alpha_5(f_x^2+f_y^2)+4\alpha_7(f_x^2+f_y^2)^2 = \theta \tag{16}$$

$$(2\alpha_2+\alpha_3\ +4(\alpha_4+\alpha_6)(f_x^2+f_y^2)+8\alpha_8(f_x^2+f_y^2)^2 = \psi, \tag{17}$$

where θ and ψ are functions of $f_x^2+f_y^2$, since all the α's are functions of the same quantity.

With these substitutions, equations (13), (15) become

$$\varrho X + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(\psi f_x^2) + \frac{\partial}{\partial y}(\psi f_x f_y) \tag{18}$$

$$\varrho Y + \frac{\partial p}{\partial y} = \frac{\partial}{\partial x}(\psi f_x f_y) + \frac{\partial}{\partial y}(\psi f_y^2) \tag{19}$$

$$\varrho Z + \frac{\partial p}{\partial z} = \frac{\partial}{\partial x}(\theta f_x) + \frac{\partial}{\partial y}(\theta f_y) \tag{20}$$

But these are precisely the equations of motion obtained by Rivlin and Green for the motion of non-Newtonian fluids characterised by the coefficients θ and ψ of viscosity and cross-viscosity respectively. Thus we find that the motion of visco-elastic flows characterised by (7) is identical with that of non-Newtonian flows for which θ and ψ are given by (16) and (17). In other words to write the equations of motion for the visco-elastic liquids, we write these for non-Newtonian liquids and replace α_1 and α_3 by

$$\alpha_1 + 2\alpha_5(f_x^2 + f_y^2) + 4\alpha_7(f_x^2 + f_y^2)^2$$

and $$(2\alpha_2 + \alpha_3) + 4(\alpha_4 + \alpha_6)(f_x^2 + f_y^2) + 8\alpha_8(f_x^2 + f_y^2)^2$$

respectively. In writing the expressions for the components of stress, the same substitutions can be used except that in the expression for t_{33} the original expression for α_3 is to be retained.

4. Rectilinear motion under uniform pressure gradient. From the above identification and from the results of Green and Erickson, we deduce that a purely rectilinear flow can be maintained by a constant pressure gradient if

(*i*) $\psi = k\theta$ *i.e.* in particular if

$$\frac{2\alpha_2 + \alpha_3}{\alpha_1} = \frac{2(\alpha_4 + \alpha_6)}{\alpha_5} = \frac{2\alpha_8}{\alpha_7} = k \tag{21}$$

This condition is obviously satisfied if all α's are constant.

or (*ii*) f is a function of a linear combination of x and y

or (*ii*) f is a function of $(x^2 + y^2)^{\frac{1}{2}}$

Discussion of other simple flows. Rivlin (1955) has discussed certain simple flows of visco-elastic fluids characterised by (7). It can be shown by arguments similar to those used above that all those flows can be deduced from those of ordinary non-Newtonian flows by certain substitutions. We give below these substitutions as well as the modifications in the components of the stress tensor.

(*a*) *Rectilinear laminar flow*: Let the velocity components be given by

$$v_1 = kx_2, \quad v_2 = 0, \quad v_3 = 0, \tag{22}$$

then the substitutions are

$$\alpha_1 \text{ by } \alpha_1 + 2\alpha_5 k^2 + 4\alpha_7 k^4 \tag{23}$$

$$\alpha_3 \text{ by } (2\alpha_2 + \alpha_3) + 4(\alpha_4 + \alpha_6)k^2 + 8\alpha_8 k^4 \tag{24}$$

In calculating t_{11}, however, the original value of α_3 is to be retained.

(b) *Torsional flow of a cylindrical mass of a fluid.* Let the velocity components be given by

$$v_r = 0, \quad v_\theta = \psi r z, \quad v_z = 0, \tag{25}$$

then the substitutions are

$$\alpha_1 \text{ by } \alpha_1 + 2\alpha_5 r^2\psi^2 + 4\alpha_7 r^4\psi^4 \tag{26}$$

$$\alpha_3 \text{ by } (2\alpha_2 + \alpha_3) + 4(\alpha_4 + \alpha_6)r^2\psi^2 + 8\alpha_8 r^4\psi^4 \tag{27}$$

Here only for $t_{\theta\theta}$ the original value are to be retained.

(c) *Helical flow in an annular space*

Here $$v_r = 0, \quad v_\theta = wr, \quad v_z = u, \tag{28}$$

and the substitutions are

$$\alpha_1 \text{ by } \alpha_1 + 2\alpha_5(r^2w'^2 + u'^2) + 4\alpha_7(u'^2 + r^2w'^2)^2 \tag{29}$$

$$\alpha_3 \text{ by } (2\alpha_2 + \alpha_3) + 4(\alpha_4 + \alpha_6)(r^2w'^2 + u'^2) + 8\alpha_8(u'^2 + r^2w'^2)^2 \tag{30}$$

In calculating the stress components t_{rr}, t_{rv}, t_{rz} we use these substitutions, while in calculating the other components we retain the original values.

6. **Dissipation of energy.** For finding this, we calculate the expression

$$D = t_{ij}\frac{\partial u_i}{\partial x_j}$$

in each case we get

(a) for rectilinear motion between planes:

$$D = k^2(\alpha_1 + 2\alpha_5 k^2 + 4\alpha_7 k^4) \tag{31}$$

(b) for torsional flow between two parallel discks

$$D = \psi^2 r^2(\alpha_1 + 2\alpha_5\psi^2 r^2 + 4\alpha_7\psi^4 r^4 \tag{32}$$

(c) for helical flow in the annular region between two coaxial cylinders.

$$D = (r^2w'^2 + u'^2)[\alpha_1 - 2\alpha_5(r^2w'^2 + u'^2) + 4\alpha_7(r^2w'^2 + u'^2)] \tag{33}$$

(d) for flow in a tube

$$D = (f_x^2 + f_y^2)[\alpha_1 + 2\alpha_5(f_x^2 + f_y^2) + 4\alpha_7(f_x^2 + f_y^2)^2] \tag{34}$$

Since the dissipation cannot be negative, we get that in the four cases discussed above, the coefficients α_1, α_5, α_7 should be such function of the invariants that

$$\left.\begin{aligned} &\alpha_1 + 2\alpha_5 k^2 + 4\alpha_7 k^4 \geq 0\,; \;\; \alpha_1 + 2\alpha_5\psi^2 r^2 + 4\alpha_7\psi^4 r^4 \geq 0\,; \\ &\alpha_1 + 2\alpha_5(u'^2 + r^2w'^2) + 4\alpha_7(u'^2 + r^2w'^2)^2 \geq 0, \\ &\alpha_1 + 2\alpha_5(f_x^2 + f_y^2) + 4\alpha_7(f_x^2 + f_y^2)^2 \geq 0 \end{aligned}\right\} \tag{35}$$

It is interesting to note that the expressions which we require to be non-negative are those corresponding to the coefficient of viscosity in the substitutions obtained above. Also the similarity in the above expressions is interesting.

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ON GENERALISED HANKEL-TRANSFORM—III

By

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1. Agarwal (1950) introduced the generalised Hankel-transform

$$g(y) = (\tfrac{1}{2})^{\lambda} \int^{\infty} (xy)^{\lambda+\frac{1}{2}} J_{\lambda}^{\mu}(\tfrac{1}{4}x^2y^2) f(x)dx \tag{1}$$

and obtained certain inversion formulae (1950, 1953) and the fundamental properties (1951) for it. Stankovic (1955) obtained the general form of functions self-reciprocal under this transform. I have in a series of papers studied the recurrence relations (1954, 1955), integral representations (1957), infinite series expansions (1957), connections with other transforms (1956, 1957), convergence theorem (1959), etc. of the transform

$$g(y) = \int_{0}^{\infty} (xy)^{\nu} J_{\lambda}^{\mu}(xy) f(x)dx, \tag{2}$$

where $J_{\lambda}^{\mu}(x)$ is the Bessel-Maitland function defined by

$$J_{\lambda}^{\mu}(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!\,\Gamma(1+\lambda+\mu r)}, \quad \mu > 0. \tag{3}$$

In view of the interesting nature of this generalised Hankel-transform, I give herewith certain rules for it. Some of these rules are similar to those of the Hankel-transform while the analogues of others do not exist in the Hankel-transform theory. I have also given some examples to illustrate their use.

In what follows, we shall call $g(y)$ the generalised Hankel-transform of $f(x)$ or $J_{\lambda,\nu}^{\mu}$-transform of $f(x)$ when they are connected by the relation (2).

2. If the $J_{\lambda,\nu}^{\mu}$-transform of $f(x)$ is $g(y)$, then we have the following rules:—

(1) $J_{\lambda,\nu}^{\mu}$*-transform of* $f(ax)$ *is* $\dfrac{1}{a} g(y/a)$.

(2) $J_{\lambda,\nu}^{\mu}$*-transform of* $xf(x)$

$$= (1/\mu y)[(\lambda-\mu)g(y;\ \lambda-\mu,\mu,\nu)-g(y;\ \lambda-\mu-1,\mu\ \nu)],$$
$$= (1/\lambda y)[g(y;\ \lambda-1,\mu,\nu+1)+\mu g(y;\lambda+\mu,\mu,\nu+2)],$$
$$= (1/\lambda)[(\lambda+1)g(y;\lambda+1\ \mu,\nu+1)-\mu g(y;\lambda+\mu+1,\mu,\nu+2)].$$

(3) $\overset{\mu}{J}_{\lambda,\nu}$-*transform of* $\frac{1}{x}f(x)$

$$= (y/\lambda)[g(y;\ \lambda-1,\mu,\nu-1)+\mu g(y;\ \lambda+\mu,\mu,\nu)]$$
$$= (y/\mu)[(\lambda-\mu)g(y;\ \lambda-\mu,\mu,\nu-2)-g(y;\ \lambda-\mu-1,\mu,\nu-2)],$$
$$= y[(\lambda+1\ g(y;\ \lambda+1,\mu,\nu-1)-\mu g(y;\ \lambda+1+\mu,\mu,\nu)].$$

The rules (2) and (3) are obvious transformations of the following result due to Wright (1933):—

$$\varrho z\varphi_{\beta+\rho} = \varphi_{\beta-1}+(1-\beta)\varphi_\beta, \tag{A}$$

where
$$\varphi_\beta = \varphi(\varrho,\beta\ ;\ z) = \sum_{l=0}^{\infty}\frac{z^l}{\Gamma(l+1)\Gamma(\varrho l+\beta)},$$

and hence
$$\overset{\mu}{J}_\lambda(x) = \varphi(\mu,1+\lambda\ ;\ -x),\quad \mu>0.$$

(4) $\overset{\mu}{J}_{\lambda,\nu}$-*transform of* $x^m f(x)$ *is* $y^{-m}g(y;\ \lambda,\mu,\nu+m)$.

(5) $\overset{\mu}{J}_{\lambda,\nu}$-*transformo f* $x^m f(x)$ *is* $(-)^m y^\nu \frac{d^m}{dy^m}[y^{-\nu}g(y;\ \lambda-m\mu,\mu,\nu)]$,

provided m is any positive integer including zero.

For, if $g(y;\lambda-m\mu,\mu,\nu) = \int_0^\infty (xy)^\nu \overset{\mu}{J}_{\lambda-m\mu}(xy)f(x)$,

then, by virtue of the relation (Wright, 1933)

$$(d/dz)\varphi_\beta = \varphi_{\beta+\rho}, \tag{B}$$

$$\frac{d}{dy}\left\{y^{-\nu}g(y;\ \lambda-m\mu,\mu,\nu)\right\} = -\int_0^\infty x^\nu \overset{\mu}{J}_{\lambda-m\mu+\mu}(xy)xf(x)dx,$$

whence, on generalisation, we get the required result.

(6) $\overset{\mu}{J}_{\lambda,}$ -*transform of* $\frac{1}{x}f(x)$ *is* $y^\nu\int_y^\infty \eta^{-\nu}g(\eta;\ \lambda+\mu,\mu,\nu)d\eta$,

provided $0<\mu\leq 1$ with an additional condition $R(\lambda+\mu)>-\frac{1}{2}$ in case $\mu=1$.

For $$y^\nu\int_y^\infty \eta^{-\nu}g(\eta;\lambda+\mu,\mu,\nu)d\eta = y^\nu\int_0^\infty x^\nu f(x)\int_y^\infty J_{\lambda+\mu}(x\eta)d\eta dx,$$

$$= \int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\,\frac{1}{x}f(x)dx.$$

(7) $J^\mu_{\lambda,\nu}$*-transform of* $\frac{1}{x}f(x^2)$

$$= \frac{y^\nu}{\pi^{\frac{1}{2}}2^{(\nu/6)+\frac{1}{2}}}\int_0^\infty \eta^{-\nu/2}e^{-\nu^2/8\eta}D_{\frac{1}{3}\nu-1}(y/(2\eta)^{\frac{1}{2}})(g(\eta\,;\ \lambda-\tfrac{1}{3}\nu\mu+2\mu, 2\mu, \tfrac{1}{3}\nu)d\eta.$$

Since
$$\begin{aligned} f(y) &= (2y)^{-\frac{1}{2}\nu}e^{-a^2/8y}D_{\nu-1}(a/(2y)^{\frac{1}{2}}),\\ &\doteqdot (\pi/2)^{\frac{1}{2}}p^{\frac{1}{2}\nu-1}e^{-ap^{\frac{1}{2}}},\\ &= \varphi(p)\end{aligned}$$

and therefore

$$\begin{aligned} p^{-\lambda-1}\varphi(p^{-\mu}) &= (\pi/2)^{\frac{1}{2}}p^{-\lambda-\frac{1}{2}\mu\nu+\mu-1}e^{-ap^{-\frac{1}{2}\mu}},\\ &\doteqdot (\pi/2)^{\frac{1}{2}}x^{\lambda+\frac{1}{2}\mu\nu-\mu}J^{\frac{1}{2}\mu}_{\lambda+\frac{1}{2}\mu\nu-\mu}(ax^{\frac{1}{2}\mu}),\end{aligned}$$

provided $\mathbf{R}(\lambda+\frac{1}{2}\mu\nu-\mu) > -1$.

Therefore (Kumar, 1953)

$$\int_0^\infty J^\mu_\lambda(xy)(2y)^{-\frac{1}{2}\nu}e^{-a^2/8y}D_{\nu-1}(a/(2y)^{\frac{1}{2}})dy$$

$$= (\pi/2)^{\frac{1}{2}}x^{\frac{1}{2}\nu-1}J^{\frac{1}{2}\mu}_{\lambda+\frac{1}{2}\mu\nu-\mu}(ax^{\frac{1}{2}}). \qquad \text{(C)}$$

The required result can now be obtained by proceeding as in (6).

(8) $J^\mu_{\lambda,\nu}$*-transform of* $x^{((\lambda+1)/\mu)-\nu-1}\int_x^\infty \xi^{\nu-(\lambda+c)/\mu}(\xi^{1/\mu}-x^{1/\mu})^{c-1}f(\xi)d\xi$

is $\mu\Gamma(c)g(y,\lambda+c,\mu,\nu)$.

For
$$\int_0^\infty (xy)^\nu J^\mu_\lambda(xy)\Big\{x^{((\lambda+1)/\mu)-\nu-1}\int_x^\infty \xi^{\nu-(\lambda+c)/\mu}(\xi^{1/\mu})-x^{1/\mu})^{c-1}f(\xi)d\xi\Big\}dx,$$

$$= y^\nu\int_0^\infty \xi^{\nu-(\lambda+c)/\mu}f(\xi)\int_0^\xi x^{((\lambda+1)/\mu)-1}(\xi^{1/\mu}-x^{1/\mu})^{c-1}J^\mu_\lambda(xy)dxd\xi,$$

$$= \mu\Gamma(c)\int_0^\infty (\xi y)^\nu J^\mu_{\lambda+c}(y\xi)f(\xi)d\xi,$$

$$= \mu\Gamma(c)g(y\,;\ \lambda+c,\mu,\nu).$$

The x-integral has been evaluated by term integration, a process easily justifiable.

(9) *$J^{\mu}_{\lambda,\nu}$-transform of* $x^{-\nu}\int_0^x\int_0^{\xi_m}\ldots\ldots\int_0^{\xi_2}\int_0^{\xi_1}\xi^{\nu-m}f(\xi)d\xi d\xi_1 d\xi_2\ldots d\xi_m$

is $g(y;\ \lambda-m\mu,\mu,\nu-m)$, where m is any positive integer including 0. For, if we write

$$F_1(\xi_m)=\int_0^{\xi_m}\int_0^{\xi_{m-1}}\ldots\ldots\int_0^{\xi_2}\int_0^{\xi_1}\xi^{\nu-m}f(\xi)d\xi d\xi_1 d\xi_2\ldots\ldots\ldots d\xi_{m-1},$$

then
$$\int_0^{\infty}(xy)^{\nu}J^{\mu}_{\lambda}(xy)x^{-\nu}\int_0^x F_1(\xi_m)d\xi_m dx,$$

$$=y^{\nu}\int_0^{\infty}F_1(\xi_m)\int_{\xi_m}^{\infty}J^{\mu}_{\lambda}(xy)dx d\xi_m$$

$$=y^{\nu-1}\int_0^{\infty}J^{\mu}_{\lambda-\mu}(y\xi_m)F_1(\xi_m)d\xi_m.$$

By repeating the above process, the required result is obtained.

3. We shall now find out the rules of finding out the $J^{\mu}_{\lambda,\nu}$-transform of the function involving the differential co efficients of $f(x)$.

(1) *$J^{\mu}_{\lambda,\nu}$-transform of* $f'(x)=y\,[g(y;\ \lambda+\mu,\mu,\nu)-\nu g(y;\ \lambda,\mu,\nu-1]$,

provided $x^{\nu}J^{\mu}_{\lambda}(xy)f(x)$ *tends to zero both as* $x\to 0$ *and as* $x\to\infty$.

This can easily be obtained on integrating by parts and then making use of 2(A).

The $J^{\mu}_{\lambda,\nu}$-transform of higher derivatives can be obtained by the repeated application of the above result. For example,

$$\begin{aligned}g''(y;\lambda,\mu,\nu)&=y\{g'(y;\ \lambda+\mu,\mu,\nu)-\nu g'(y;\ \lambda,\mu,\nu-1)\}\\&=y^2\{g(y;\ \lambda+2\mu,\mu,\nu)-2\nu g(y;\ \lambda+\mu,\mu,\nu-1)\\&\qquad+\nu(\nu-1)g(y;\ \lambda,\mu,\nu-2)\}.\end{aligned}$$

(2) *$J^{\mu}_{\lambda,\nu}$-transform of* $\left(x^{-\nu}\dfrac{d}{dx}\right)^m\{x^{\nu}f(x)\}=y^m g(y;\ \lambda+m\mu\ \mu,\nu)$,

provided $x^{\nu}J^{\mu}_{\lambda}(xy)\left(x^{-\nu}\dfrac{d}{dx}\right)^m\{x^{\nu}f(x)\}$ *vanishes both at zero and at* ∞.

For
$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\left(x^{-\nu}\frac{d}{dx}\right)\{x^\nu f(x)\}dx$$
$$= y\int_0^\infty (xy)^\nu J_{\lambda+\mu}^\mu(xy)f(x)dx,$$

by virtue of the relation 2(B) and the conditions of the rule. Hence, on generalisation, we obtain the required result.

(3) $J_{\lambda,\nu}^\mu$*-transform of* $\dfrac{d^2f}{dx^2}+\dfrac{\nu}{x}\dfrac{df}{dx}$

$$= y^2[g(y\,;\ \lambda+2\mu,\mu,\nu)-\nu g(y\,;\ \lambda,\mu,\nu-1)],$$

provided $x^\nu J_\lambda^\mu(xy)f'(x)$ *and* $x^\nu J_\lambda^\mu(xy)f(x)$ *vanish both at 0 and at* ∞.

For
$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\,\frac{d^2f}{dx^2}\,dx = -y^\nu\int_0^\infty \{\nu x^{\nu-1}J_\lambda^\mu(xy)-x^\nu yJ_{\lambda+\mu}^\mu(xy)\}\frac{df}{dx}dx.$$

Therefore
$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\left\{\frac{d^2f}{dx^2}+\frac{\nu}{x}\frac{df}{dx}\right\}dx = y\int_0^\infty (xy)^\nu J_{\lambda+\mu}^\mu(xy)\frac{df}{dx}dx,$$
$$= y^2[g(y\,;\ \lambda+2\mu,\mu,\nu)-\nu g(x\,;\ \lambda+\mu,\mu,\nu-1)].$$

(4) The rules given in sections 2 and 3 can be employed to find out $J_{\lambda,\nu}^\mu$-transforms of certain functions and to evaluate certain integrals. As an illustration, we give below few examples.

Example. *1.* For $\nu=1$, 2(G) gives

$$\int_0^\infty J_\lambda^\mu(xy)x^{-\frac{1}{2}}e^{-a^2/4x}dx = \pi^{\frac{1}{2}}y^{-\frac{1}{2}}J_{\lambda-\frac{1}{2}\mu}^{\frac{1}{2}\mu}(ay^{\frac{1}{2}}).$$

Therefore, by rule 6, we have

$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)x^{-\nu-3/2}e^{-a^2/4x}dx = \pi^{\frac{1}{2}}y^\nu\int_y^\infty \eta^{-\frac{1}{2}}J_{\lambda+\frac{1}{2}\mu}^{\frac{1}{2}\mu}(a\eta^{\frac{1}{2}})d\eta,$$
$$= 2\pi^{\frac{1}{2}}y^\nu\int_{y^{\frac{1}{2}}}^\infty J_{\lambda+\frac{1}{2}\mu}^{\frac{1}{2}\mu}(a\xi)d\xi,$$
$$= \frac{2(\pi)^{\frac{1}{2}}y^\nu}{a}J_\lambda^{\frac{1}{2}\mu}(ay^{\frac{1}{2}}).$$

Hence $J^{\mu}_{\lambda,\nu}$-transform of $x^{-\nu-3/2}e^{-a^2/4x}$ is $\dfrac{2\pi^{\frac{1}{2}}}{a}\, y^{\nu} J^{\frac{1}{2}\mu}_{\lambda}(ay^{\frac{1}{2}})$

and
$$\int_0^{\infty} J^{\mu}_{\lambda}(xy)x^{-3/2}e^{-a^2/4x}dx = \frac{2\pi^{\frac{1}{2}}}{a} J^{\frac{1}{2}\mu}_{\lambda}(ay^{\frac{1}{2}}),$$

provided $0 < \mu \leq 1$ with an additional condition $\mathbf{R}(\lambda) > -3/2$.

Example 2. Since
$$\int_0^{\infty} (xy)^{\nu} J^{\mu}_{\lambda}(xy)x^{-\nu}dx = y^{\nu-1} J^{\mu}_{\lambda-\mu}(ay),$$

then, by rule (5), we have
$$\int_0^{\infty} x^m J^{\mu}_{\lambda}(xy)dx = (-)^m \frac{d^m}{dy^m}\left[\frac{1}{y} J^{\mu}_{\lambda-\overline{m+1}\mu}(ay)\right],$$

where m is any positive integer including zero and $0 < \mu \leq 1$ with an additional condition $\mathbf{R}(\lambda) > 2m + 3/2$.

Example 3. From 2(C), by the help of the rule 5, we have
$$\int_0^{\infty} J^{\mu}_{\lambda}(xy)x^{m-\frac{1}{2}\nu}e^{-a^2/8x}D_{\nu-1}(a/(2x)^{\frac{1}{2}})dx$$
$$= (-)^m 2^{\frac{1}{2}\nu-\frac{1}{2}}(\pi)^{\frac{1}{2}} \frac{d^m}{dy^m}\left[y^{\frac{1}{2}\nu-1} J^{\frac{1}{2}\mu}_{\lambda+\frac{1}{2}\nu\mu-\overline{m+1}\mu}(ay^{\frac{1}{2}})\right],$$

where m is any positive integer including 0 and $0 < \mu < 1$.

Example. 4. If we take $f(x) = 1$, then the $J^{\mu}_{\lambda,\nu}$-transform of
$$\frac{1}{x} f(x^2) = \frac{\Gamma(\nu)}{(1+\lambda-\mu\nu)}$$

on making use the following result due to Gupta (1948):—
$$\int_0^{\infty} (xy)^{\nu} J^{\mu}_{\lambda}(xy)dx = \frac{\Gamma(1+\nu)}{\Gamma(1+\lambda-\mu-\mu\nu)} \cdot \frac{1}{y},$$

valid when $0 < \mu \leq 1$ and $\mathbf{R}(\nu) > -1$ with an additional condition $\mathbf{R}(\lambda - 2\nu) > -\frac{1}{2}$ in case $\mu = 1$.

Also
$$g\left(\eta;\ \lambda - \frac{1}{3}\nu\mu + 2\mu, 2\mu; \frac{\nu}{3}\right) = \int_0^{\infty} (x\eta)^{\nu/3} J^{2\mu}_{\lambda-\frac{1}{3}\nu\mu+2\mu}(x\eta)dx,$$
$$= \frac{\Gamma(1+\frac{1}{3}\nu)}{\Gamma(1+\lambda-\mu\nu)} \cdot \frac{1}{\eta}.$$

Hence, by rule 7, we have

$$\int_0^\infty \eta^{-\frac{1}{2}\nu/2-1} e^{-y^2/8\eta} D_{\frac{1}{2}\nu-1}(y/(2\eta)^{\frac{1}{2}})\,d\eta = \frac{\Gamma(\nu)}{\Gamma(\frac{1}{2}\nu+1)} \cdot \frac{\pi^{\frac{1}{2}} 2^{\nu/6+\frac{1}{2}}}{y^\nu},$$

valid for $R(\nu) > \frac{1}{2}$.

Example 5. With the help of the integral (Kumar, 1955)

$$\int_0^\infty (xy)^{n+\frac{1}{2}m} J^2_{2n+m-c+1}(xy) e^{-y} He_m(2y^{\frac{1}{2}})\,dy$$

$$= \pi^{\frac{1}{2}} 2^{c-n-\frac{1}{2}m-\frac{1}{2}} x^{\frac{1}{2}c-\frac{1}{2}} \left(x^{\frac{1}{2}} \frac{d}{dx}\right)^c \left\{x^{\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2}} J_{n+\frac{1}{2}}(x^{\frac{1}{2}})\right\},$$

where m, n and c are any positive integers including zero, and the rule 8, it can be easily shown that

$$\int_0^\infty x^{n+\frac{1}{2}m-\frac{1}{2}c} J^2_{2n+m-c+1}(xy) \int_x^\infty \xi^{-\frac{1}{2}}(\xi^{\frac{1}{2}}-x^{\frac{1}{2}})^{c-1} e^{-\xi} He_m(2\xi)^{\frac{1}{2}}\,d\xi\,dx$$

$$= 2^{-n-\frac{1}{2}m+\frac{1}{2}} \Gamma(c) \pi^{\frac{1}{2}} y^{-\frac{1}{2}n-\frac{1}{2}} J_{n+\frac{1}{2}}(y^{\frac{1}{2}}).$$

This result can be easily verified by changing the order of integration and then evaluating the ξ-integral.

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SOME FUNDAMENTAL CONSIDERATIONS ABOUT MASS IN GENERAL AND ABOUT NEGATIVE MASSES

By

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Abstract. Every particle can be assigned three numbers representing its three masses *viz*, the inertial, the active gravitational and the passive gravitational. In Section (1) we consider qualitatively the motion of two particles due to their mutual attraction or repulsion when their masses can be positive or negative and indicate twenty different possibilities In Section (2) the motion of test particles and of light pulses in Schwarzschild's external field and in the external field of a matter dipole has been considered The new possibilities regarding the nature of mass considered here lead to unexpected consequences regarding the velocities of test particles and light pulses in a gravitational field. It has been shown that the velocity of a light pulse exceeds the fundamental velocity c ($=1$ in natural units) and is not bounded above. The test particles, while they always move with velocities less than the velocities of light at the points of their instantaneous location, acquire velocity maxima which are also unbounded above.

1. In classical physics the concept of mass is introduced in two different contexts, firstly in the second law of motion

$$P = mf, \tag{1.1}$$

and secondly in the law of gravitation

$$F = \gamma \frac{m_1 m_2}{r^2}, \tag{1.2}$$

While the mass m appearing in equation (1.1) measures the capacity of the body to resist motion due to a force acting on it, the masses m_1 and m_2 appearing in equation (1.2) measure the capacities of the two bodies, separated by a distance r, to attract or get attracted. We thus arrive at three distinct notions of mass which can explicitly be stated as follows.

(A) *The inertial mass of a body is a measure of its capacity to resist motion when a force acts on it*

(B) *The active gravitational mass of a body is a measure of its capacity to attract another body.*

(C) *The passive gravitational mass of a body is a measure of its capacity to respond to the attraction of another body.*

If we distinguish the three masses associated with a body by giving suffixes i, a,

and p, the equations (1.1) and (1.2) become

$$P = m_i f, \tag{1.3}$$

$$F_{12} = \gamma \frac{m_{1a} m_{2p}}{r^2}, \tag{1.4A}$$

$$F_{21} = \gamma \frac{m_{1p} m_{2a}}{r^2}, \tag{1.4B}$$

where F_{ij} measures the force of attraction on the body j due to the body i.

The accelerations experienced by the two bodies due to their mutual attraction wil be given by

$$f_1 = \frac{F_{21}}{m_{1i}} = \frac{\gamma}{r^2} m_{2a} \frac{m_{1p}}{m_{1i}} \tag{1.5}$$

$$f_2 = \frac{F_{12}}{m_{2i}} = \frac{\gamma}{r^2} m_{1a} \frac{m_{2p}}{m_{2i}} \tag{1.6}$$

We observe that (1.5) and (1.6) involve only the ratios of the passive gravitational to the inertial masses of the bodies.

The units having been suitably chosen Newton's third law of motion implies the equality of the active and passive gravitational masses of a body which immediately follows from equations (1.4A) and (1.4B). The equality of the inertial and the gravitational masses of a body is axiomatic. Thus in classical physics the three masses of a body are represented by the same number.

General relativity does not make any distinction between the active and passive gravitational masses. The equality of the inertial and the gravitational masses is postulated in the principle of equivalence which plays a fundamental role in the formulation of general relativity. Experimentally the ratio is found to differ from unity by a term of the order of 10^{-8}. The gravitational mass appears for the first time as a constant of integration in Schwarzschild's exterior solution. This mass is essentially the active gravitational mass. The inertial and the passive gravitational masses do not appear anywhere explicitly. However, we can still give meaning to these terms. In the operational definition of the field *viz.*,

$$\textit{Field intensity} = \underset{\textit{mass} \to 0}{\mathrm{Lt.}} \frac{\textit{Force experienced}}{\textit{mass}}, \tag{1.7}$$

the mass appearing is necessarily the passive gravitational mass. The inertial mass can be given the same meaning as in equation (1.1).

The force appearing in equation (1.7) thus involves the inertial mass so that we get on the right hand side of equation (1.7) only a ratio of the inertial to the passive gravitational mass. Now the principle of equivalence demands that the ratio of the inertial to

the passive gravitational mass should be the same for all bodies, and that in a way explains why these two masses do not explicitly appear in the equations of motion in general relativity.

Recently the possibility of negative masses has been suggested and discussed by Bondi (1957). A number of possibilities arise for the accelerations of two bodies under their mutual attraction or repulsion. We give below tables indicating the different possibilities on the basis of equations (1.5) and (1.6) which could be taken as a guide for the corresponding relativistic problems. The ratio of inertial to passive gravitational mass is taken to be $+1$ or -1. 'Towards' appearing in column 'Body 1' will mean towards body 2 and so on.

TABLE I

$m_i/m_p = +1$

	MASS						ACCELERATION	
	BODY 1			BODY 2			BODY 1	BODY 2
S. No.	Inertial	Passive gravitational	Active gravitational	Inertial	Passive gravitational	Active gravitational		
1.	+	+	+	+	+	+	Towards	Towards
2.	+	+	+	+	+	−	Away	Towards
3.	+	+	+	−	−	+	Towards	Towards
4.	+	+	+	−	−	−	Away	Towards
5.	+	+	−	+	+	−	Away	Away
6.	+	+	−	−	−	+	Towards	Away
7.	+	+	−	−	−	−	Away	Away
8.	−	−	+	−	−	+	Towards	Towards
9.	−	−	+	−	−	−	Away	Towards
10.	−	−	−	−	−	−	Away	Away

TABLE II

$m_i/m_p = -1$

	MASS						ACCELERATION	
	BODY 1			BODY 2			BODY 1	BODY 2
S. No.	Inertial	Passive gravitational	Active gravitational	Inertial	Passive gravitational			
1.	+	−	+	+	−	+	Away	Away
2.	+	−	+	−	+	+	Away	Away
3.	+	−	+	+	−	−	Towards	Away
4.	+	−	+	−	+	−	Towards	Away
5.	−	+	+	−	+	+	Away	Away
6.	−	+	+	+	−	−	Towards	Away
7.	−	+	+	−	+	−	Towards	Away
8.	+	−	−	+	−	−	Towards	Towards
9.	+	−	−	−	+	−	Towards	Towards
10.	−	+	−	−	+	−	Towards	Towards

We observe that for the ratio $m_i/m_p = +1$, a body is always accelerated towards a positive active mass and away from a negative active mass, as observed by Bondi but for the ratio $m_i/m_p = -1$, a body is always accelerated towards a negative active mass and away from a positive active mass.

For each ratio m_i/m_p only two of the three masses of a body can be given any of the two signs independently. Hence there are in all $2 \times 2 = 4$ possible combinations of signs for the three masses of a body. Therefore, the number of possible combinations of signs for the two bodies will be $4 \times 4 = 16$. The number of distinct cases is reduced to 10 on account of symmetry.

2. We now investigate the situations arising out of importing a negative active gravitational mass within the frame-work of general relativity. In Schwarzschild's exterior solution if we replace m by $-m$ we get

$$ds^2 = -\left(1+\frac{2m}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + \left(1+\frac{2m}{r}\right)dt^2. \tag{2.1}$$

The metric is obviously regular everywhere except at the point $r=0$. For the radial motion of a test particle in this field, following the method indicated by Singh and Pandey in a recent communication, we obtain from the equations of the geodesics

$$\ddot{r} = -\frac{3m}{r^2}\left(1+\frac{2m}{r}\right)^{-1}\dot{r}^2 + \frac{m}{r^2}\left(1+\frac{2m}{r}\right) \tag{2.2}$$

where a dot overhead indicates a differentiation with respect to time. The first integral of this equation gives

$$\dot{r}^2 = \frac{2m}{a}\left(1+\frac{2m}{a}\right)^{-1}\left(1+\frac{2m}{r}\right)^2\left(1-\frac{a}{r}\right), \tag{2.3}$$

where $\dot{r}=0$ for $r=a$. Equation (2.3) shows that for $\dot{r}$ to be real r has always to be greater than a.

It is easy to see that $\dot{r}$ reaches a maximum for

$$r = \frac{6ma}{4m-a}, \tag{2.4}$$

and this maximum velocity is given by

$$\dot{r}_{max} = \frac{2}{3.3^{\frac{1}{2}}}\left(1+\frac{2m}{a}\right), \tag{2.5}$$

which is greater than unity for

$$a < \frac{4m}{3.3^{\frac{1}{2}}-2}$$

and does not have an upper bound. Thus if a test particle starts from a point

$$r = a < \frac{4m}{3.3^{\frac{1}{2}}-2}$$

it will aquire a velocity greater than unity and the maximum velocity will tend to infinity as a tends to zero.

Equation (2.4) also shows that the test particle attains a velocity-maximum only if it starts from a position $r = a < 4m$. For the body at the origin to have a mass equal in magnitude to that of the sun this will require the numerical value of its volume density to be greater than 2.10^{15} gms/c.c. which is unusually high.

For the radial propagation of light we have to put

$$ds = 0 \text{ with } \dot{\theta} = \dot{\varphi} = 0,$$

so that from equation (2.1) we get the radial velocity $\dot{r}'$ of light to be

$$\dot{r}' = 1 + \frac{2m}{r}, \tag{2.6}$$

which is greater than one for all values of r and is unbounded above. However, equations (2.3) and (2.6) show that the radial velocity of light pulses is greater than the radial velocity of a test particle for all r.

Similar results can be obtained for Bondi's general relativistic model of an uniformly accelerated matter dipole. The metric for this model in a co-moving frame of reference is

$$ds^2 = e^{2\varphi}dt^2 - e^{-2\varphi+2\sigma}(dr^2 + dz^2) - r^2 e^{-2\varphi}d\theta^2, \tag{2.7}$$

where
$$\varphi = \tfrac{1}{2}\log\{[r^2+z^2]^{\frac{1}{2}}+z\} - \frac{m_1}{\{r^2+(z-h_1)^2\}^{\frac{1}{2}}} - \frac{m_2}{\{r^2+(z-h_2)^2\}^{\frac{1}{2}}}, \tag{2.8}$$

$$\sigma = \tfrac{1}{2}\log\{\tfrac{1}{2} + \tfrac{1}{2}z(r^2+z^2)^{-\frac{1}{2}}\}, \tag{2.9}$$

for points external to the material distributions. m_1 and m_2 are the masses of the two spheres of radii a_1 and a_2 placed with their centres on the z-axis at $z = h_1$ and $z = h_2$ respectively, h_1 and h_2 $(h_2 > h_1)$ being positive. For $a_1 << h_1$, $a_1 << (h_2 - h_1)$ and $a_2 << (h_2 - h_1)$ we have

$$\left.\begin{aligned} m_1 &\approx -\frac{(h_2-h_1)^2}{2h_2}, \\ m_2 &\approx \frac{(h_2-h_1)^2}{2h_1}. \end{aligned}\right\} \tag{2.10}$$

The singularities of the metric are located on the z-axis at $z = h_1$, $z = h_2$ and $z \leq 0$, the last one being purely a coordinate singularity.

For this metric the equations of the geodesics are

$$\frac{d^2r}{ds^2}+\left(\frac{\partial\sigma}{\partial r}-\frac{\partial\varphi}{\partial r}\right)\left\{\left(\frac{dr}{ds}\right)^2-\left(\frac{dz}{ds}\right)^2\right\}+\left(r^2\frac{\partial\varphi}{\partial r}-r\right)e^{-2\sigma}\left(\frac{d\theta}{ds}\right)^2$$

$$+e^{4\varphi-2\sigma}\frac{\partial\varphi}{\partial r}\left(\frac{dt}{ds}\right)^2+2\left(\frac{\partial\sigma}{\partial z}-\frac{\partial\varphi}{\partial z}\right)\frac{dr}{ds}\frac{dz}{ds}=0, \tag{2.11}$$

$$\frac{d^2\theta}{ds^2}+2\left(\frac{1}{r}-\frac{\partial\varphi}{\partial r}\right)\frac{dr}{ds}\frac{d\theta}{ds}-2\frac{\partial\varphi}{\partial z}\frac{dz}{ds}\frac{d\theta}{ds}=0, \tag{2.12}$$

$$\frac{d^2z}{ds^2}+\left(\frac{\partial\varphi}{\partial z}-\frac{\partial\sigma}{\partial z}\right)\left\{\left(\frac{dr}{ds}\right)^2-\left(\frac{dz}{ds}\right)^2\right\}+r^2e^{-2\sigma}\frac{\partial\varphi}{\partial z}\left(\frac{d\theta}{ds}\right)^2$$

$$+e^{4\varphi-2\sigma}\frac{\partial\varphi}{\partial z}\left(\frac{dt}{ds}\right)^3+2\left(\frac{\partial\sigma}{\partial r}-\frac{\partial\varphi}{\partial r}\right)\frac{dr}{ds}\frac{dz}{ds}=0, \tag{2.13}$$

and

$$\frac{d^2t}{ds^2}+2\frac{\partial\varphi}{\partial r}\frac{dr}{ds}\frac{dt}{ds}+2\frac{\partial\varphi}{\partial z}\frac{dz}{ds}\frac{dt}{ds}=0. \tag{2.14}$$

If initially

$$\theta=\alpha,\ \frac{d\theta}{ds}=0,$$

we get from equation (2.12)

$$\frac{d^2\theta}{ds^2}=0.$$

Hence motion is restricted to the plane $\theta=\alpha$.

Now equation (2.14) can be written as

$$\frac{d^2t}{ds^2}+2\frac{d\varphi}{ds}\frac{dt}{ds}=0, \tag{2.15}$$

which gives as its first integral

$$\frac{dt}{ds}=Ae^{-2\varphi}. \tag{2.16}$$

With the help of equation (2.15) we can reduce equations (2.11) and (2.12) to

$$\ddot{r}-2r\ddot{\varphi}+\left(\frac{\partial\sigma}{\partial r}-\frac{\partial\varphi}{\partial r}\right)(\dot{r}^2-\dot{z}^2)+e^{4\varphi-2\sigma}\frac{\partial\varphi}{\partial r}+2\left(\frac{\partial\sigma}{\partial z}-\frac{\partial\varphi}{\partial z}\right)\dot{r}\dot{z}=0, \tag{2.17}$$

and

$$\ddot{z}-2\dot{z}\dot{\varphi}+\left(\frac{\partial\varphi}{\partial z}-\frac{\partial\sigma}{\partial z}\right)(\dot{r}^2-\dot{z}^2)+e^{4\varphi-2\sigma}\frac{\partial\varphi}{\partial z}+2\left(\frac{\partial\sigma}{\partial r}-\frac{\partial\varphi}{\partial r}\right)\dot{r}\dot{z}=0. \tag{2.18}$$

Now for $r=0$ it can easily be seen that

$$\frac{\partial\varphi}{\partial r}=\frac{\partial\sigma}{\partial r}=0.$$

Therefore, if we put $\dot{r} = 0$ when $r = a$ as an initial condition equation (2.17) will give

$$\ddot{r} = 0.$$

Equation (2.18) then gives the equation of motion of a test particle along the z-axis which can be simplified and integrated to give an expression for $\dot{z}^2$. However, $\dot{z}^2$ is determined more easily from the metric (2.7) with the help of equation (2.16) as

$$\dot{z}^2 = 4z^2 \exp\left\{-\frac{4\alpha z}{(z-h_1)(z-h_2)}\right\}\left[1-\frac{z}{a}\exp\left\{-2\alpha\frac{(a-z)(az-h_1h_2)}{(a-h_1)(a-h_2)(z-h_1)(z-h_2)}\right\}\right] \tag{2.19}$$

where we have put $\dot{z} = 0$ for $z = a$ and $\alpha = \dfrac{(h_2-h_1)^3}{2h_1h_2}$.

Let a be greater than h_2. Then for $\dot{z}$ to be real z has always to be less than a. A test particle, therefore, falls from a position $z = a > h_2$ towards the dipole and reaches its maximum velocity $\dot{z}_{\max}$ given by

$$\dot{z}_{\max} = \frac{2}{3^{\frac{1}{2}}}\,\bar{z}\,\exp\left\{-\frac{2\alpha\bar{z}}{(\bar{z}-h_1)(\bar{z}-h_2)}\right\} \tag{2.20}$$

at a point $\bar{z}$ such that

$$\frac{\bar{z}}{a}\exp\left\{-\frac{\alpha(a-\bar{z})(a\bar{z}-h_1h_2)}{(a-h_1)(a-h_2)(\bar{z}-h_1)(\bar{z}-h_2)}\right\} = \frac{2}{3}.$$

For

$$a,\ z > h_2$$

$$\frac{\bar{z}}{a} > \frac{2}{3}$$

or

$$\bar{z} > \frac{2a}{3}.$$

Equation (2.20) clearly shows that $\dot{z}_{\max}$ does not have an upper bound‡

For light the velocity $\dot{z}'$ is found to be given by

$$\dot{z}' = 2z\,\exp\left\{-\frac{2\alpha z}{(z-h_1)(z-h_2)}\right\} \tag{2.21}$$

‡ It is an inherent property of the uniformly accelerated metric $ds^2 = z^2dt^2-dx^2-dy^2-dz^2$ of which (2.7) is a transform for $m_1=m_2=0$ that test particles and light pulses do not have an upper bound for their velocities in the space-time described by it.

which is also obviously unbounded above. In this case again equations (2.19) and (2.21) show that the particle velocity is less than the velocity of light at any point along the axis of z.

The equations of the geodesics ignore entirely the considerations of mass of a test particle. The treatment given above, remains valid when for the test particle the ratio $m_i/m_p = +1$. The case $m_i/m_p = -1$ and also the case when $m_a \neq 0$ can be included only if the problem is solved as a two body problem so as to take account of the masses of both the bodies. This requires that the problem of n bodies be solved in general relativity by deriving the equations of motion from the field equations themselves in the general case of three types of finite masses associated with each body.

The above investigation shows that the introduction of negative masses within the frame-work of general relativity contradicts the fundamental hypothesis regarding the velocity of light. It needs a careful examination as to how the concept of negative mass can be accommodated in a relativity theory.

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A NEW GENERALIZATION OF THE LAPLACE TRANSFORM

By

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1. Introduction. The integral equation

$$\varphi(p) = p\int_0^\infty e^{-pt}f(t)dt,\ R(p) > 0, \tag{1.1}$$

represents the classical Laplace Transform and the functions $\varphi(p)$ and $f(t)$, related by (1.1), are said to be operationally related to each other. $\varphi(p)$ is called the image and $f(t)$, the original. Symbolically we write

$$\varphi(p) \doteqdot f(t) \text{ or } f(t) \doteqdot \varphi(p). \tag{1.2}$$

Many generalizations of the aforesaid transform have been given from time to time by different mathematicians like Meijer (1940, 1941), Boas (1942), Varma (1947, 1951) and others. In the present note we take up a new generalization on the lines of Meijer and Varma. We give some theorems analogous to the well-known Goldstein's theorem (Goldstein, 1932) and also establish the Uniqueness theorem and an Inversion Formula.

2. We consider the integral relation

$$\varphi(p) = p\int_0^\infty e^{-\frac{1}{2}pt}W_{k+\frac{1}{2},m}(pt)(pt)^{-\lambda-\frac{1}{2}}f(t)dt,\ R(p) > 0, \tag{2.1}$$

provided of course, the integral on the right exists. Here also we call $\varphi(p)$, the image and $f(t)$, the original and symbolically we write

$$\varphi(p)\ \underset{\underset{m}{\longleftarrow}}{\overset{\overset{k}{\longrightarrow}}{\lambda}}\ f(t) \text{ or } f(t)\ \underset{\underset{m}{\longleftarrow}}{\overset{\overset{k}{\longrightarrow}}{\lambda}}\ \varphi(p) \tag{2.2}$$

It is evident that (2.1) reduces to the Meijer transform if we take

$$\lambda = k, \tag{2.3}$$

and to the generalization due to Varma if we assume

$$\lambda = -m. \tag{2.4}$$

Further if we take $$\lambda = k = \pm m, \tag{2.5}$$

we fall back upon the original Laplace transform (1.1).

3. The Uniqueness Theorem. In case of the Laplace transform it has been established by Lerch that the function $\varphi(p)$ is the image of a unique function $f(t)$. Before

establishing a similar theorem in connection with the present generalization we prove the following lemma.

Lemma. *If*
$$\int_0^\infty e^{-\frac{1}{2}pt} W_{k,m}(pt) t^{\mu} F(t)\,dt = 0, \tag{3.1}$$
then
$$F(t) \equiv 0, \tag{3.2}$$
provided that

(*i*) $F(t)$ *is continuous in* $t \geq 0$,

(*ii*) $F(t) = 0(t^{\delta})$, $R(\mu \pm m + \delta) > -\frac{3}{2}$, *when* t *is small and*

(*iii*) $R(k-m) > \frac{1}{2}$.

Proof. We multiply (3.1) by $p^{m-\frac{1}{2}}(p-z)^{k-m-3/2}$ and integrate with respect to p between the limits (z, ∞), $z > 0$

Then we obtain
$$\int_z^\infty p^{m-\frac{1}{2}}(p-z)^{k-m-3/2}dp \int_0^\infty e^{-\frac{1}{2}pt} W_{k,m}(pt) t^{\mu} F(t)\,dt = 0,$$
or
$$\int_0^\infty t^{\mu} F(t)\,dt \int_z^\infty e^{-\frac{1}{2}pt} W_{k,m}(pt) p^{m-\frac{1}{2}}(p-z)^{k-m-3/2}dp = 0, \tag{3.3}$$
the change in the order of integrations being justifiable by absolute convergence of the integrals under the conditions imposed.

Now evaluating the p-integral, we obtain
$$\Gamma(k-m-\tfrac{1}{2}) Z^{k+m-\frac{1}{2}} \int_0^\infty e^{-zt} t^{\mu+m+\frac{1}{2}} F(t)\,dt = 0,$$
$$\int_0^\infty e^{-zt} t^{\mu+m+\frac{1}{2}} F(t)\,dt = 0. \tag{3.4}$$
Hence by Lerch's theorem $F(t) \equiv 0$, (Lerch, 1903).

Thus having established the above lemma, the proof of the Uniqueness Theorem, stated below, follows as a direct consequence.

Theorem 1. *Let* $f_1(t)$ *and* $f_2(t)$ *be continuous in* $t \geq 0$

and
$$f_1(t) \;\underset{\longleftarrow}{\overset{\overset{k}{\longrightarrow}}{\lambda}}\; \varphi(p), \tag{3.5}$$
and also
$$f_2(t) \;\underset{\underset{m}{\longleftarrow}}{\overset{\overset{k}{\longrightarrow}}{\lambda}}\; \varphi(p), \tag{3.6}$$
Then
$$f_1(t) \equiv f_2(t). \tag{3.7}$$

4. In connection with the Laplace transform, Goldstein (1932) has proved the theorem :

"If $$\varphi_1(p) \doteqdot f_1(t), \tag{4.1}$$

and $$\varphi_2(p) \doteqdot f_2(t), \tag{4.2}$$

then $$\int_0^\infty \varphi_1(t) f_2(t) \frac{dt}{t} = \int_0^\infty \varphi_2(t) f_1(t) \frac{dt}{t} \tag{4.3}$$

provided the changes in the orders of integrations involved are justifiable."

An analogue of the above theorem in the present case may be given as follows.

Theorem 2. *Let $f_1(t)$ and $f_2(t)$ be continuous in $t \geq 0$,*

and $$\varphi_1(p) \underset{\underset{m}{\longleftarrow}}{\overset{\overset{k}{\longrightarrow}}{\lambda}} f_1(t), \tag{4.4}$$

and $$\varphi_2(p) \underset{\underset{m}{\longleftarrow}}{\overset{\overset{k}{\longrightarrow}}{\lambda}} f_2(t). \tag{4.5}$$

Then $$\int_0^\infty \varphi_1(t) f_2(t) \frac{dt}{t} = \int_0^\infty \varphi_2(t) f_1(t) \frac{dt}{t}, \tag{4.6}$$

provided the integrals converge and

(*i*) $f_1(t) = 0(t^{\delta_1})$, $R(\delta_1 \pm m - \lambda) > -1$, *when t is small,*

and (*ii*) $f_2(t) = 0(t^{\delta_2})$, $R(\delta_2 \pm m - \lambda) > -1$, *when t is small.*

The proof is quite simple, being similar to that of the Goldstein's theorem, and hence we omit.

Below we give two more theorems analogous to the above theorem but slightly more general.

Theorem 3. *Let $f_1(t)$ and $f_2(t)$ be continuous in $t \geq 0$,*

and $$\varphi_1(p) \underset{\underset{m}{\longleftarrow}}{\overset{\overset{k}{\longrightarrow}}{\lambda_1}} f_1(t), \tag{4.7}$$

and $$\varphi_2(p) \underset{\underset{m}{\longleftarrow}}{\overset{\overset{k}{\longrightarrow}}{\lambda_2}} f_2(t). \tag{4.8}$$

Then $$\int_0^\infty \varphi_1(t) f_2(t) t^{\lambda_1 - \lambda_2 - 1} dt = \int_0^\infty \varphi_2(t) f_1(t) t^{\lambda_2 - \lambda_1 - 1} dt, \tag{4.9}$$

provided the integrals converge and

$$\left.\begin{array}{ll} (i) & f_1(t) = 0(t^{\delta_1}),\ R(\delta_1 \pm m - \lambda_1) > -1 \\ \textit{and} \quad (ii) & f_2(t) = 0(t^{\delta_2}),\ R(\delta_2 \pm m - \lambda_2) > -1 \end{array}\right\} \quad \textit{when } t \textit{ is small.}$$

Proof. We have

$$I = \int_0^\infty \varphi_1(t) f_2(t) t^{\lambda_1-\lambda_2-1} dt$$

$$= \int_0^\infty f_2(t) t^{\lambda_1-\lambda_2} dt \int_0^\infty e^{-\frac{1}{2}yt} W_{k+\frac{1}{2},m}(yt)(yt)^{-\lambda_1-\frac{1}{2}} f_1(y) dy, \tag{4.10}$$

by (4 7).

Now changing the order of integrations, which is justifiable under the conditions imposed upon $f_1(t)$ and $f_2(t)$, we get

$$I = \int_0^\infty f_1(y) y^{\lambda_2-\lambda_1} dy \int_0^\infty e^{-\frac{1}{2}yt} W_{k+\frac{1}{2},m}(yt)(yt)^{-\lambda_2-\frac{1}{2}} f_2(t) dt,$$

$$= \int_0^\infty \varphi_2(y) f_1(y) y^{\lambda_2-\lambda_1-1} dy, \tag{4.11}$$

by (4.8). Hence the theorem

Evidently theorem 2. follows from theorem 3., if we take $\lambda_1 = \lambda_2$.

Theorem 4. *Let $f_1(t)$ and $f_2(t)$ be continuous in $t \geq 0$,*

and

$$\varphi_1(p) \underset{\overleftarrow{m_1}}{\overset{\overrightarrow{k_1}}{\lambda_1}} f_1(t), \tag{4.12}$$

and

$$\varphi_2(p) \underset{\overleftarrow{m_2}}{\overset{\overrightarrow{k_2}}{\lambda_2}} f_2(t). \tag{4.13}$$

Then

$$\int_0^\infty f_1(zt)\varphi_2(t)\frac{dt}{t} \underset{\overleftarrow{m_1}}{\overset{\overrightarrow{k_1}}{\lambda_1}} \varphi(p) \underset{\overleftarrow{m_2}}{\overset{\overrightarrow{k_2}}{\lambda_2}} \int_0^\infty f_2(zt)\varphi_1(t)\frac{dt}{t}, \tag{4.14}$$

provided that $\varphi(p)$ exists and

$$\left.\begin{array}{ll} (i) & f_1(t) = 0(t^{\delta_1}),\ R(\delta_1 \pm m_1 - \lambda_1) > -1 \\ (ii) & f_2(t) = 0(t^{\delta_2}),\ R(\delta_2 \pm m_2 - \lambda_2) > -1 \end{array}\right\} \quad \textit{when } t \textit{ is small},$$

and (iii) $\varphi_1(t)t^{\delta_2-1}$ *and* $\varphi_2(t)t^{\delta_1-1}$ *belong to* $L(0,\infty)$.

Proof. Let us consider the integral

$$\varphi'(p) = p\int_0^\infty e^{-\frac{1}{2}pz} W_{k_1+\frac{1}{2},m_1}(pz)(pz)^{-\lambda_1-\frac{1}{2}} dz \int_0^\infty f_1(zt)\varphi_2(t)\frac{dt}{t}$$

$$= p\int_0^\infty \varphi_2(t)\frac{dt}{t}\int_0^\infty e^{-\frac{1}{2}pz} W_{k_1+\frac{1}{2},m_1}(pz)(pz)^{-\lambda_1-\frac{1}{2}} f_1(zt) dz, \tag{4.15}$$

the change in the order of integrations being justifiable under the conditions imposed upon $f_1(t)$ and $\varphi_2(t)$.

Hence, in view of (4.12), we obtain

$$\varphi'(p) = \int_0^\infty \varphi_2(t)\varphi_1(p/t)\frac{dt}{t}. \tag{4.16}$$

Similarly we obtain

$$\varphi''(p) = p\int_0^\infty e^{-\frac{1}{2}pz}W_{k_2+\frac{1}{2},m_2}(pz)(pz)^{-\lambda_2-\frac{1}{2}}dz\int_0^\infty f_2(zt)\varphi_1(t)\frac{dt}{t}$$
$$= \int_0^\infty \varphi_1(t)\varphi_2(p/t)\frac{dt}{t}. \tag{4.17}$$

Now writing p/t for t in (4 17) we obtain (4.16). Hence we conclude

$$\varphi'(p) = \varphi''(p) = \varphi(p)(say).$$

Evidently theorem 2. follows from theorem 4. by making use of the Uniqueness theorem (theorem 1.) proved above, if we take

$$k_1 = k_2,\ \lambda_1 = \lambda_2,\ m_1 = m_2.$$

5. **Examples.** (1) Let
$$f_1(t) = t^{\alpha},\ R(\alpha\pm m-\lambda_1) > -1, \tag{5.1}$$
and
$$f_2(t) = t^{\lambda_1+k-1}e^{-\beta/t} \tag{5.2}$$

Then, according to theorem 3,
$$\varphi_1(p) = \frac{\Gamma_*(\alpha\pm m-\lambda_1+1)}{\Gamma(\alpha-k-\lambda_1+1)}\,p^{-\alpha},\ddagger \tag{5.3}$$

and
$$\varphi_2(p) = 2\beta^{\kappa}p^{1-\lambda_2}k_{2m}(2(\beta p)^{\frac{1}{2}}). \tag{5.4}$$

Hence
$$\int_0 \varphi_1(t)f_2(t)t^{\lambda_1-\lambda_2-1}dt = \Gamma_*(\alpha\pm m-\lambda_1+1)\beta^{\kappa-\lambda_1-\alpha-1}, \tag{5.5}$$

and
$$\int_0^\infty \varphi_2(t)f_1(t)t^{\lambda_2-\lambda_1-1}dt = \Gamma_*(\alpha\pm m-\lambda_1+1)\beta^{\kappa-\lambda_1-\alpha-1}, \tag{5.6}$$

which is true according to theorem 3.

(2). Let $f_1(t) = t^{\alpha}$, $R(\alpha\pm m-\lambda_1) > -1$ and $f_2(t) = t^{\mu}e^{-\beta t}$, $R(\mu\pm m-\lambda_2) > -1$. (5.7)

Then according to theorem 3,
$$\varphi_1(p) = \frac{\Gamma_*(\alpha\pm m-\lambda_1+1)}{\Gamma(\alpha-k-\lambda_1+1)}\,p^{-\alpha}, \tag{5.8}$$

and
$$\varphi_2(p) = \frac{\Gamma_*(\mu\pm m-\lambda_2+1)}{\Gamma(\mu-k-\lambda_2+1)}\,\frac{p^{m-\lambda_2+1}}{(\beta+p)^{\mu+m-\lambda_2+1}}\,{}_2F_1\left(\begin{matrix}\mu+m-\lambda_2+1, m-k\,; \\ \mu-k-\lambda_2+1\,;\end{matrix}\ \frac{\beta}{\beta+p}\right)$$
$$R(p+2\beta) > 0. \tag{5.9}$$

Hence
$$\int_0^\infty \varphi_2(t)f_1(t)t^{\lambda_2-\lambda_1-1}dt$$

‡ $\Gamma_*(A\pm B)\equiv\Gamma(A+B)\Gamma(A-B)$.

$$=\frac{\Gamma_*(\mu\pm m-\lambda_2+1)}{\Gamma(\mu-k-\lambda_2+1)}\int_0^\infty t^{\alpha+m-\lambda_1}\,{}_2F_1\left(\begin{matrix}\mu+m-\lambda_2+1,m-k\,;\\ \mu-k-\lambda_2+1\,;\end{matrix}\;\frac{\beta}{\beta+t}\right)\frac{dt}{(\beta+t)^{\mu+m-\lambda_2+1}}$$

$$=\frac{\Gamma_*(\mu\pm m-\lambda_2+1)}{\Gamma(\mu-k-\lambda_2+1)}\beta^{\alpha+\lambda_2-\lambda_1-\mu}\int_0^1 (1-t)^{\alpha+m-\lambda_1}\,{}_2F_1\left(\begin{matrix}\mu+m-\lambda_2+1,m-k\,;\\ \mu-k-\lambda_2+1\,;\end{matrix}\;t\right)\frac{dt}{t^{\alpha+\lambda_2-\lambda_1-\mu+1}}$$

$$=\frac{\Gamma(\mu-m-\lambda_2+1)\Gamma(\alpha+m-\lambda_1+1)\Gamma(\mu+\lambda_1-\lambda_2-\alpha)}{\Gamma(\mu-k-\lambda_2+1)\beta^{\mu+\lambda_1-\lambda_2-\alpha}}\,{}_2F_1\left(\begin{matrix}\mu+\lambda_1-\lambda_2-\alpha,m-k\,;\\ \mu-k-\lambda_2+1\,;\end{matrix}\;1\right)$$

$$R(\mu+\lambda_1-\lambda_2-\alpha)>0.$$

$$=\frac{\Gamma_*(\alpha\pm m-\lambda_1+1)\Gamma(\mu+\lambda_1-\lambda_2-\alpha)}{\Gamma(\alpha-k-\lambda_1+1)\beta^{\mu+\lambda_1-\lambda_2-\alpha}},\tag{5.10}$$

and
$$\int_0^\infty \varphi_1(t)f_2(t)t^{\lambda_1-\lambda_2-1}dt=\frac{\Gamma_*(\alpha\pm m-\lambda_1+1)}{\Gamma(\alpha-k-\lambda_1+1)}\int_0^\infty e^{-\beta t}t^{\mu+\lambda_1-\lambda_2-\alpha-1}dt$$

$$=\frac{\Gamma_*(\alpha\pm m-\lambda_1+1)\Gamma(\mu+\lambda_1-\lambda_2-\alpha)}{\Gamma(\alpha-k-\lambda_1+1)\,\beta^{\mu+\lambda_1-\lambda_2-\alpha}},\tag{5.11}$$

which is again true according to theorem 3.

(3). Let $\quad f_1(t)=t^\alpha$ and $f_2(t)=t^{\lambda_2+k_2-1}e^{-\beta t^{-1}}$ $R(\alpha\pm m_1-\lambda_1)>1.$ (5.12)

Then according to theorem 4.,
$$\varphi_1(p)=\frac{\Gamma_*(\alpha\pm m_1-\lambda_1+1)}{\Gamma(\alpha-k_1-\lambda_1+1)}p^{-\alpha},\tag{5.13}$$

and
$$\varphi_2(p)=2\beta^{k_2}p^{1-\lambda_2}k_{2m_2}(2(\beta p)^{\frac12}).\tag{5.14}$$

Hence
$$f'(z)=\int_0^\infty f_1(zt)\varphi_2(t)\frac{dt}{t}$$

$$=2\beta^{k_2}z^\alpha\int_0^\infty k_{2m_2}(2(\beta t)^{\frac12})t^{\alpha-\lambda_2}dt,$$

$$=\Gamma_*(\alpha\pm m_2-\lambda_2+1)z^\alpha\beta^{k_2+\lambda_2-\alpha-1},\ R(\alpha-\lambda_2+1)>|R(m_2)|\tag{5.15}$$

and
$$f''(z)=\int_0^\infty f_2(zt)\varphi_1(t)\frac{dt}{t}$$

$$=\frac{\Gamma_*(\alpha\pm m_1-\lambda_1+1)}{\Gamma(\alpha-k_1-\lambda_1+1)}z^{\lambda_2+k_2-1}\int_0^\infty e^{-\beta/zt}t^{\lambda_2+k_2-\alpha-2}dt$$

$$=\frac{\Gamma_*(\alpha\pm m_1-\lambda_1+1)}{\Gamma(\alpha-k_1-\lambda_1+1)}z^\alpha\beta^{\lambda_2+k_2-\alpha-1}\int_0^\infty e^{-t}t^{\alpha-k_2-\lambda_2}dt$$

$$=\frac{\Gamma_*(\alpha\pm m_1-\lambda_1+1)\Gamma(\alpha-k_2-\lambda_2+1)z^\alpha}{\Gamma(\alpha-k_1-\lambda_1+1)\beta^{\alpha-k_2-\lambda_2+1}}.R(\alpha-k_2-\lambda_2)\geq-1.\tag{5.16}$$

Now we obtain

$$\varphi'(p) = p\int_0^\infty e^{-\frac{1}{2}pz} W_{k_1+\frac{1}{2},m_1}(pz)(pz)^{-\lambda_1-\frac{1}{2}} f'(z)\,dz$$

$$=\Gamma_*(\alpha \pm m_2-\lambda_2+1)p^{\frac{1}{2}-\lambda_1}\beta^{k_2+\lambda_2-\alpha-1}\int_0^\infty e^{-\frac{1}{2}pz} W_{k_1+\frac{1}{2},m_1}(pz) z^{\alpha-\lambda_2-\frac{1}{2}}\,dz$$

$$=\frac{\Gamma_*(\alpha\pm m_1-\lambda_1+1)\Gamma_*(\alpha\pm m_2-\lambda_2+1)}{\Gamma(\alpha-k_1-\lambda_1+1)\beta^{\alpha-k_2-\lambda_2+1}p^{\alpha}} \tag{5.17}$$

and
$$\varphi''(p) = p\int_0^\infty e^{-\frac{1}{2}pz} W_{k_2+\frac{1}{2},m_2}(pz)(pz)^{-\lambda_2-\frac{1}{2}} f'(z)\,dz$$

$$=\frac{\Gamma_*(\alpha\pm m_1-\lambda_1+1)\Gamma(\alpha-k_2-\lambda_2+1)}{\Gamma(\alpha-k_1-\lambda_1+1)\beta^{\alpha-k_2-\lambda_2+1}p^{\lambda_2-\frac{1}{2}}}\int_0^\infty e^{-\frac{1}{2}pz} W_{k_2+\frac{1}{2},m_2}(pz) z^{\alpha-\lambda_2+\frac{1}{2}}\,dz,$$

$$=\frac{\Gamma_*(\alpha\pm m_1-\lambda_1+1)\Gamma_*(\alpha\pm m_2-\lambda_2+1)}{\Gamma(\alpha-k_1-\lambda_1+1)\beta^{\alpha-k_2-\lambda_2+1}p^{\alpha}}, \tag{5.18}$$

$$=\varphi'(p), \text{ by (5.17).}$$

6. An Inversion Formula. We have

$$\varphi(p) = p\int_0^\infty e^{-\frac{1}{2}pt} W_{k+\frac{1}{2},m}(pt)(pt)^{-\lambda-\frac{1}{2}} f(t)\,dt. \tag{6.1}$$

Let us assume that $\Phi(s)$, the Mellin transform of $\varphi(p)$, exists. Then

$$\Phi(s) = \int_0^\infty p^s dp\int_0^\infty e^{-\frac{1}{2}pt} W_{k+\frac{1}{2},m}(pt)(pt)^{-\lambda-\frac{1}{2}} f(t)\,dt,\ s=\sigma+i\tau,$$

$$=\int_0^\infty t^{-\lambda-\frac{1}{2}}f(t)\,dt\int_0^\infty e^{-\frac{1}{2}pt}W_{k+\frac{1}{2},m}(pt)p^{s-\lambda-\frac{1}{2}}\,dp, \tag{6.2}$$

the change in the order of integrations being justifiable if we assume that $t^{-\sigma-1}f(t)$ belongs to $L(0, \infty)$ and $R(s\pm m-\lambda)>-1$.

Hence evaluating the p-integral, we obtain

$$\Phi(s) = \frac{\Gamma_*(s\pm m-\lambda+1)}{\Gamma(s-k-\lambda+1)}\int_0^\infty f(t)t^{-s-1}\,dt$$

or
$$\int_0^\infty f(t)t^{-s-1}\,dt = \frac{\Gamma(s-k-\lambda+1)}{\Gamma_*(s\pm m-\lambda+1)}\Phi(s). \tag{6.3}$$

Now applying the Mellin's inversion formula, we get

$$\tfrac{1}{2}[f(t+0)+f(t-0)] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(s-k-\lambda+1)}{\Gamma_*(s\pm m-\lambda+1)}\Phi(s)t^s\,ds,\ s=\sigma+i\tau. \tag{6.4}$$

$$\sigma > R(k+\lambda-1).$$

Hence the theorem:

Theorem 6. *Let $f(t)$ be bounded in $t \geq 0$, and of bounded variation in the neighbourhood of the point t,*

and
$$\varphi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} W_{k+\frac{1}{2},m}(pt)(pt)^{-\lambda-\frac{1}{2}} f(t)\,dt.$$

Then
$$\tfrac{1}{2}[f(t+0)+f(t-0)] = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s-k-\lambda+1)}{\Gamma_*(s\pm m-\lambda+1)}\Phi(s)t^s ds,\ s = \sigma+i\tau,$$
$$\sigma > R(k+\lambda-1),$$

where $\Phi(s)$ is the Mellin transform of $\varphi(p)$,

provided that $f(t)t^{-\sigma-1}$ *and* $\varphi(t)t^{\sigma-1}$ *belong to* $L(0, \infty)$,

and $R(s\pm m-\lambda) > -1$.

Further if we take $f(t)$ to be continuous in $t \geq 0$, we obtain the inversion formula in the form

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s-k-\lambda+1)}{\Gamma_*(s\pm m-\lambda+1)}\Phi(s)t^s ds, \tag{6.5}$$

provided the other conditions of the theorem are satisfied.

We may verify the inversion formula by considering an example here.

We know that

$$\frac{\Gamma_*(\mu\pm m-\lambda+1)}{\Gamma(\mu-k-\lambda+1)} p^{-\mu} e^{-p^{-1}} = \int_0^\infty e^{-\frac{1}{2}pt} W_{k+\frac{1}{2},m}(pt)(pt)^{-\lambda-\frac{1}{2}} t^\mu \times {}_1F_2\left(\begin{matrix}\mu-k-\lambda+1\,; \\ \mu\pm m-\lambda+1\,;\end{matrix}\ -t\right)dt\ \ddagger \tag{6.6}$$

Then according to the inversion formula, established above, we have

$$\frac{\Gamma(\mu-k-\lambda+1)}{\Gamma_*(\mu\pm m-\lambda+1)} t^\mu {}_1F_2\left(\begin{matrix}\mu-k-\lambda+1\,; \\ \mu\pm m-\lambda+1\,;\end{matrix}\ -t\right)$$
$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s-k-\lambda+1)}{\Gamma(s\pm m-\lambda+1)}\Gamma(\mu-s)t^s ds, \tag{6.7}$$
$$R(\mu) > \sigma > R(k+\lambda-1),$$

‡ ${}_1F_2(\alpha;\ \beta\pm\gamma;\ -t) \equiv {}_1F_2(\alpha;\ \beta+\gamma,\ \beta-\gamma;\ -t)$.

which may be shown to be true by evaluating the integral by swinging round the contour so as to enclose the poles on the right of the contour).

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BOOK REVIEW

S. Mitra and G. K. Dutt :—*A Textbook of the Integral Calculus*, W. Heffer & Sons Ltd., Cambridge, Great Britain, 1959, xiv + 393 pp. 30s.

The book is supposed to cover the B. Sc. Syllabus of the Indian Universities. The authors wished to deal with definite integrals first but the position in India, as they remark, prevented them from doing so. We do not know whether the authors studied in the Calcutta University. Even as early as the year 1916, the only text book followed here was *Integral Calculus for Beginners* by J. Edwards (MacMillan & Co. Ltd) which begins with the Integral as the limit of a sum.

In establishing the uniqueness of the Indefinite Integrals (p. 2) the authors have referred to § 58 of their text of the Differential Calculus, where the discussion given is not sufficient.

The authors want to develop Integral Calculus in a logical manner, unfortunately they assume the existence of area without proceeding to explain the difficulties involved in defining area enclosed by a closed curve (p. 72).

In defining surface integrals, no mention is made of the surface of Schwarz (p. 268).

The authors have used Fourier series in partial differential equations before treating Fourier series.

The way the development of Fourier Series has been done is too difficult to be followed by an average student.

However, there is no major flaw in the treatment of the subject.

Lastly the price of the book is beyond the reach of the most of the Indian students.

The printing and the binding of the book are good.

B. DUTT.

CALCUTTA MATHEMATICAL SOCIETY.

Report of the Council on the state of affairs of the Society for the year 1960, placed in the Annual General Meeting of the Society.

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the state of affairs of the Society for the year 1960 as required by the provisions of Rule 25 of the Society's Constitution.

The Council. The council of the Society for the year 1960 consisting of officers and other members elected at the last Annual General meeting with Assistant Secretary and Editorial Secretary was constituted as follows:

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Ram Behari,
N. G. Shabde,
M. C. Chaki,
Saktikanta Chakraborty, (*Asst. Secretary.*)
B. C. Chatterji,
A. C. Choudhury,
S. C. Das Gupta,
B. R. Seth,
B. S. Ray,

General. In this year, in addition to usual activities, the Society has undertaken some new ventures viz. the renovation of the library of the Society, the organisation of Symposia on recent advances in different branches of Mathematics, the organisation of Symposia on teaching of Mathematics in Schools to devise ways and means for improving the standard of teaching of mathematics in India, the arrangement of publication of the short report of the progress of the researches in Mathematics in India in the last fifty years. The Society could not undertake a few more important works viz. the arrangement of publication of the Proceedings of the Symposia on Fluid

Mechanics, the organisation of the Symposia on teaching of Mathematics in the Graduate and Post-Graduate level, the increase in the number of subscribed journals in the Society. A short report of the activities of the Council in different heads is given below:

Symposia. (1) *Symposium on Teaching of Mathematics in Schools.* On the 28th., 29th. and 30th. of March, 1960 a symposium on Teaching on Mathematics in the schools was held. Prof. S K. Chakraborty presided over the General Symposium. Principal S. K.Basu, Prof. D. N. Ray and Prof. S. K. Chakraborty presided over discussions for Higher Secondary stage, Middle stage and Primary stage respectively. Many eminent Mathematicians and distinguished teachers from Colleges and Schools participated in the symposium. A short report of the proceedings of the Symposium will be published soon.

(II). *Symposium on Teaching of Fluid Mechanics in Post Graduate and Research level.* On the 23rd. of September, 1960 a Symposium on teaching of Fluid Mechanics in Post-Graduate and Research level was held. Prof. N. R. Sen presided over the Symposium. Prof. N. L. Ghosh, Dr. J. N. Kapur and Dr. A. K. Roy made important suggestions. The report is being made ready for publication.

(III). *Symposium on Fluid Mechanics.* On the 22nd., 23rd.and 24th. of September 1960 a Symposium on Fluid Mechanics was held. Prof. N. R. Sen presided over the Symposium.

Speaker		*Topic*
N. R. Sen	...	On the study of Fluid Dynamics.
N. L. Ghosh	...	On some recent investigations regarding the Drag and lift on aerofoils.
J. N. Kapur	...	Some aspects on superposability in Fluid Dynamics.
A. K. Ray	...	Study of some aspects of the Flow characteristics in shock wave boundary layer interaction problems.
S. D. Nigam & K. S. Sinha		Some problems in the theory of rotating Fluids.
M. K. Jain	...	On extremal point collocation method for Fluid flow problems.
R. S. Nanda	...	Viscous flows with suction or injection.
R. N. Bhattacharya		Bed effect on wave resistance of a ship moving in a circle.
A. R. Sen	...	Stationary phase and missi g waves.
J. N. Kapur & R. K. Jain		Parallel flow in an annular channel in hydromagnetics.
K. M. Ghosh	...	The early period decay in case of any general type Turbulence with specific hypothesis borrowed from normal distributions.

Effort is being made to publish the proceedings.

Publications. During the year under review the Society has published three issues of the Bulletin viz. Vol. 51, No. 4 and Vol 52 Nos. 1 & 2. The Council takes the opportunity to record here its indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and to the Officers and members of the Staff of the University Press for their valued services. It may, however, be mentioned that the Society bears the cost of paper and blocks and the matter of Bulletin upto May, 1960 was composed by the Society's compositors.

The Council is glad to announce that the printing of the short report of the progress of researches in Mathematics in India during the last fifty year is made possible by benevolent patronage of the Government of India and the Government of West Bengal.

The Council is glad to announce that the printing of the Proceedings of the Sympsoium on Teaching of Mathematics in Schools is in progress. The report on the Teaching of Fluid Mechanics in Post-Graduate and Research level is also being made ready.

The Council is glad to announce that the printing of the first part of the Golden Jubilee Commemoration Volume has been completed and will soon come out of the press. The printing of the second part is in progress.

Exchange of Publication. The transmission of the Society's publications to various Countries of the World has been carried on regularly during the year. The Society has received one hundred and thirtytwo journals in exchange this year.

Library. The Society has purchased only five books and subscribed sixteen journals this year. Some useful journals lying unbound for several years have been bound. The Council feels that the Society should purchase more books and journals if the library is to serve the need of the active research workers to the desired extent, which, however, is not possible for shortage of funds. The rack has been expanded and a book case has been purchased for better upkeeping of journals.

Meetings During 1960. The Council has met nine times this year and there have been six General meetings of members in which sixteen original papers have been discussed and nine original papers have been taken as read.

Membership. The Council desires to report that during the year under review twenty-one new names have been added to the list of Society's members. Two gentlemen have become life members of the Society.

Delegations to Learned Societies. In reply to invitations from the Indian Science Congress Association the Council nominated the following members to represent the Society at their 48th. Session at Roorkee.

(1) Principal S K. Basu, M.A., Ph.D., Presidency College, Calcutta-12.

(2) Dr. M. C. Chaki, M.Sc., D.Phil., Calcutta University, Calcutta.

(3) Prof. N. L. Ghosh, M.Sc., D.Phil., Presidency College, Calcutta.

(4) Prof. M. Ray, D.Sc., Agra College, Agra.

(5) Prof. B. B. Sen, D.Sc., Jadavpur University, Jadavpur.

(6 Prof. B. R. Seth, D.Sc., Indian Institute of Technology, Kharagpur.

At the 20th. Session of the Indian Mathematical Society at Chandigarh, the Council nominated the following members to represent the Society.

(1) Dr. A. C. Choudhury, M.Sc., D.Phil , Calcutta University, Calcutta.

(2) Dr. G. Bandyopadhyay, M.Sc., D.Phil., Indian Institute of Technology, Kharagpur.

The Council nominated the following members to represent the Society at the Silver Jubilee Celebrations of the National Institute of Sciences of India at Delhi.

(1) Prof. S. N. Bose, F.R S , National Professor.

(2) Principal, S. K. Basu, Ph D., Presidency College, Calcutta-12.

(3) Prof. Ram Behari, D Sc , Delhi University, Delhi.

(4) Prof. P. L. Bhatnagar, D.Sc., Indian Institute of Sc., Bangalore.

(5) Prof. S. K. Chakraborty, D.Sc. B.E. College, Howrah.

(6) Prof. B. N. Prasad, D.Sc., Allahabad University, Allahabad.

(7) Prof. B. B. Sen, D.Sc., Jadavpur University, Calcutta-32.

(8) Prof. N. R. Sen, D.Sc., Ph.D., Calcutta.

(9) Prof. R. N. Sen, Ph D., Calcutta University, Calcutta.

(10) Prof. A. C. Banerji, Ex-Vice-Chancellor, Allahabad University.

Finance. The Annual Accounts of the Society have been presented to the Council in the standardized form by the auditors Sri P. L. Ganguli and Dr. M. Mitra. The Council offers them its sincere thanks for their honorary services. The Council also takes this opportunity to make a few comments on the audited statement of accounts during the year under review. A glance at the receipts and disbursements reveals some improvement in the financial position of the Society, thanks to the grant given by the Government of India which was not forthcoming for the last two years.

It is, however, obvious that unless the Society receives more grants-in-aid from the Government and donations from the members and the public, it will be difficult to increase the activities of the Society.

The Society received the following grants for the year under review :—

(*i*) Government of India—Rs 4,500/-

(*ii*) Government of W. Bengal—Rs. 2,000/-

(*iii*) National Institute of Sciences of India—Rs. 1,000/-

(*iv*) Govt of West Bengal (Symposium on Teaching)—Rs. 1,000/-

The Council offers its grateful thanks to the Government of India, the Government of West Bengal and the National Institute of Sciences of India for these grants.

Obituary. This year the Society lost two of its esteemed members Dr. H. M. Sengupta, Reader, Department of Pure Mathematics, Calcutta University and Dr. N. G. Shabde, Chairman, S.S.C.E. Board, M.P. The Council takes the opportunity to offer condolence to the bereaved families.

CALCUTTA MATHEMATICAL SOCIETY

Receipts and Disbursements Accounts of the Calcutta Mathematical Society for the Year Ending 31st. December, 1960.

Receipts.

		Rs. nP.	Rs. nP.
Opening Balance			
1. (a) With Secretary			
(i) In cash	4.81		
(ii) In stamps	1.25	5.56	
(b) Balance at Banks			
(i) State Bank of India (Gen. Fund)	2704.80		
(ii) Do (K. K. G. P. Fund) ...	1052.10		
(iii) Do (C. E. Cullis Fund) ...	956.00		
(iv) United Bank of India ...	2077.90		
Do (in suspense) ...	214.01		
(v) Postal Savings Bank ...	35.50	7,045.87	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000)		5,663.72	
(d) G. P, Notes (K. K. G. P. Fund) (Face value Rs. 2,000)		1,937.48	
(e) G. P. Notes (C. E. Cullis Fund) (Face value Rs. 8,000)		6,558.06	21,205.13
2. Membership subscription (including Life Membership)	1,943.12		
3. Admission fees	210.00		
4. Sale proceeds	5,336 42		
5. Donation (Received in cash) ...	115 00	7,604 54	
6. Grants :—			
(i) Govt. of West Bengal	2,000 00		
(ii) Govt. of India	4,500 0		

Disbursements

		Rs. nP.	Rs. nP.
1. Establishment			
(a) Salary	340.96		
(b) Provident Fund	84.00	424.96	
2. Meetings	..	323.03	
3. Books & Journals (including binding)			
(a) Books	500.51		
(b) Journals	2,144.79		
(c) Binding	1,347.75	3,993.05	
4. Bulletins			
(a) Establishment	1,211.88		
(b) Compositors' Salary	1,131.00		
(c) Do (amount set aside... under commitment) ...	1,250.00		
(d) Paper, Blocks, Types etc ...	2,382.30		
(e) Postage	601.28		
(f) Conveyance (Press)	44.13	6.620.59	
9. Printing & Stationery		231.66	
6. Postage (General)		301.06	
7. Bank charges		73.66	
8. Miscellaneous (including conveyance charges)		210.08	
9. Symposium on Teaching		2,071.51	
Do (under commitment)		539.20	
10. Arrear publication		1,235.01	
Do (under commitment)		764.99*	
11. Library renovation		1.314.75	
12. Cost of Type-writer (as per commitment)		1,212.00	19,345.55
13. Closing Balance			
(a) With Secretary			

7. Interest			
(a) G. P.Notes (General Fund) ...	180.00		
(b) Do (K. K. G. P. Fund) ...	60 00		
(c) Do (C. E. Cullis Fund) ...	240.00		
(d) Postal Savings Accounts ...	1.78		
		481.78	
Total ... Rs.			37,791.45

(b) Balance at Banks			
(i) State Bank of India (Gen. Fund)	1,612.45		
Less under commitment for Arrear Publication	764.99		
		847.46**	
(ii) Do (K. K. G. P. Fund) ...		1,111.60	
(iii) Do (C. E. Cullis Fund) ...		1,195.00	
(iv) United Bank of India ...	4.522.39		
Less :—			
Compositors' salary under commitment	1,250.00		
Cost of Type-writer under commitment	1,242.00		
Cheques issued in 1960 but not cashed	401.25		
Amount set aside for Symposium on Teaching	539.20		
	3,432.45		
		1,089.94**	
(v) Postal Savings (excluding P.F. 84/-)		37.28	
			4,286.64
(c) G. P. Notes (General Fund) (Face value Rs. 6,000)		5,663.72	
(d) Do (K. K. G P. Fund) (Face value Rs. 2,000)		1,937.48	
(e) Do (C. E. Cullis Fund) (Face value Rs. 8,000)		6,558.06	
			14,159.26
		Total ...Rs.	37,791.45

To

The Members of the Calcutta Mathematical Society,

We have examined the above accounts with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

* Rs. 870.53 (out of Rs 2,000) for Arrear publication is lying with the Golden Jubilee Fund of which Rs. 547.09 has been spent in 1960. A separate account for this is given in the audited statement of accounts of Golden Jubilee.

** Rs 1,500.00 earmarked for publication of the short report of the progress of research in India during the last fifty years.

Sd. P. L. Ganguli,
Sd. M. Mitra, } *Auditors*

CALCUTTA MATHEMATICAL SOCIETY

93, Acharya Prafulla Chandra Road,
Calcutta-9 (*India*)

Dated, February 27, 1961.

NOTICE

The Calcutta Mathematical Society has entered into a reciprocity agreement with the American Mathematical Society. The conditions of the agreement are given below:

Under such an agreement, any member of the Calcutta Mathematical Society may join the American Mathematical Society by submitting an application for membership on the usual form. He would continue to pay the usual dues to the Calcutta Mathemetical Society but would pay half dues only to the American Mathematical Society. At present, the full rate is $ 14.00 and reciprocating members pay annual dues of $ 7.00. Members of the Society under one of the reciprocity agreements enjoy all the usual privileges of membership such as receiving the Notices, the Proceedings, and the Bulletin and presenting papers at meetings of the Society, purchasing the publications of the Society and certain other publications at reduced rates and subscribing to Mathematical Reviews at the members' rate of $ 16.00 per year. They are also permitted to substitute the Transactions for the Proceedings upon payment of a premium to dues of $ 10.00 in 4-volume years. They are also allowed to substitute Mathematical Reviews for the Proceedings by paying a premium of $ 10.00 to the dues.

On the other hand, under such a reciprocity agreement, any member of the American Mathematical Society would be admitted to membership in the Calcutta Mathematical Society at his reqest and would pay half dues to the Calcutta Mathematical Society. He would then enjoy all the usual privileges of membership of the Calcutta Mathematical Society.

There is one exception to the above schedule of dues. If a member of our Society became a member of the American Mathematical Society through such a reciprocity agreement and then visited the U.S.A., he would pay full dues to the American Mathematical Society for the time spent in the U.S.A. A similar arrangement would take effect for a member of the American Mathematical Society visiting our country.

Members of our Society who are desirous of being members of the American Mathematical Society under reciprocity agreement should write to

The Executive Director,
American Mathematical Society
190, Hope Street,
Providence 6, Rhode Island,
U.S.A.

Application forms for membership of the American Mathematical Society may be had form the office of our Society.

P. P. CHATTARJI,
Secretary,
Calcutta Mathematical Society.

All correspondence with the Society, subscriptions to the Bulletin, admission fees and annual contributions of members are to be sent to the *Secretary, Calcutta Mathematical Society, 92, Acharya Prafulla Chandra Road, Calcutta-9.*

Papers intended for publication in the Bulletin of the Society, and all Editorial Correspondence, should be addressed to the *Editorial Secretary, Calcutta Mathematical Society.*

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office, or from its agents—Messrs. Bowes & Bowes, Booksellers and Publishers, 1, Trinity Street, Cambridge, England.

NOTICE TO AUTHORS

The manuscript of each paper communicated for publication in this Journal should be legibly written (preferably type-written) on one side of the paper and should be accompanied by a short *abstract* of the paper.

References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of paper, the author's name following form, *viz.*, name or names of authors; year of publication; name of the journal (abbreviation); number of volume; and lastly, the page number. The following would be a useful illustration:

Wilson, B. M. (1922), *Proc. Lond. Math. Soc.* (2), **21**, 235.

Authors of papers printed in the Bulletin are entitled to receive, free of cost, 50 separate copies of their communications. They can, however, by previous notice to the Secretary, ascertain whether it will be possible to obtain more copies even on payment of the usual charges.

CONTENTS

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ON LATTICOIDS

By

A. C. Choudhury, *Calcutta*

(*Received—June 1, 1960*)

1. Introduction. An upper semilattice is an additive commutative semigroup in which every element is idempotent. It can be therefore expected that many properties of an abelian group can easily be generalised to properties of an upper semilattice. One such property of an abelian group has already been generalised (Choudhury). It has been proved there that the endomappings of an upper semilattice form an upper m-semilattice. This corresponds to the wellknown result that the endomappings of an additive abelian group form a ring This fact in the theory of an abelian group produces the notion of a module and all the results associated with it. It is natural therefore to expect that the m-semilattice of the endomappings of U will generate the corresponding notions of a latticoid and many beautiful results will follow from this notion. The object of this paper is to build up the theory of a latticoid. An upper semilattice U which admits a scalar multiplication with the elements of an m-semilattice Ω has been defined to be a latticoid. The condition that a subset may be a sublatticoid has been found and a dependence has been developed. Further the isotone mappings between a right latticoid U and a two sided latticoid V have been proved to form a left latticoid and has been denoted as $\mathrm{Iso}(U,V)$. A commutative m-semilattice Ω can be considered as a two sided latticoid and consequently $U^* = \mathrm{Iso}(U,\Omega)$ is a left latticoid when U is a right latticoid. U^* may be called the dual of U. The connection between U and U^* has been determined. Further it has been shown that an isotone mapping f between two latticoids U,V over Ω induces an isotone mapping f^t between U^* and V^*. Lastly these results have used to obtain a representation theory for an upper m-semilattice.

2. Latticoid. An upper semilattice U which admits the elements of m-semilattice Ω as a right scalar multiplier is called a right latticoid if the scalar multiplier satisfies the laws:—

(*i*) $\alpha a \in U$ when $\alpha \in U$, $a \in \Omega$, $b \in \Omega$.

(*ii*) $\alpha(a+b) = \alpha a + \alpha b$, $(\alpha+\beta)a = \alpha a + \beta a$

(*iii*) $(\alpha a)b = \alpha(ab)$

(*iv*) $\alpha 1 = \alpha$ when 1 is the topmost element of U.

If the scalar multiplication is on the left *i.e.* if $a\alpha$ is defined, then the system will be called a left latticoid.

One can easily construct an example of a latticoid. For, from the elements of Ω, construct elements.

$$\alpha = (a_1, a_2, \ldots\ldots\ldots\ldots, a_n)$$

and impose on the elements α the rules :—

$$(i) \quad \alpha a = (a_1 a,\ a_2 a, \ldots\ldots\ldots, a_n a)$$

$$(ii) \quad \alpha+\beta = (a_1, b_1,\ a_2+b_2, \ldots\ldots\ldots, a_n+b_n)$$

$$\text{when } \alpha = (a_1, a_2 \ldots\ldots, a_n)$$

$$\beta = (b_1, b_2, \ldots\ldots\ldots, b_n).$$

Then the set U of elements α form a right latticoid. For, as a_i, b_i are the elements of an m-semilattice, the sum a_i+b_i are associative, commutative and idempotent. Hence U is a semilattice. The laws (i) to (iv) can be trivially verified. Hence U is a right latticoid.

Let U be a right latticoid over the m semilattice Ω. Then the mappings $D_a : \alpha \to \alpha a$ is an endomorphism of U. For $D_a(\alpha+\beta) = (\alpha+\beta)a = \alpha a+\beta a = D_a(\alpha)+D_a(\beta)$.

D_a will be called the dilation of the semilattice U. The condition $(\alpha a)b = \alpha(ab)$ can be written as $D_{ab}(\alpha) = D_b D_a(\alpha)$. Hence $D_b D_a = D_{ab}$. This gives a rule of multiplication for the dilation D_a.

Similarly the mapping $t_\alpha : a \to \alpha a$ where α is a fixed element of U, is a homomorphism of Ω into U. For $t_\alpha(a) = \alpha a$ and $t_\alpha(a+b) = \alpha(a+b) = \alpha a+\alpha b = t_\alpha(a)+t_\alpha(b)$.

Sublatticoids of a latticoid U over Ω are those subsets which are again latticoids over Ω. Then one can prove the theorem:

A subset U_1, is a sublatticoid if and only if

$$(i) \quad \alpha+\beta \in U_1, \text{ when } \alpha,\beta \in U_1$$

$$(ii) \quad \alpha a \in U_1, \text{ when } \alpha \in U_1,\ a \in \Omega.$$

Proof. If U_1 is a latticoid, then these conditions must hold by the definition. Conversely, let these conditions hold. Then if $\alpha,\beta,\gamma \in U_1$, $\alpha+\beta = \beta+\alpha$ and $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ since $\alpha,\beta,\gamma \in U$. Also as $\alpha \in U$, $\alpha+\alpha = \alpha$. Hence U_1 is a semilattice. Further as $\alpha \in U$, and $a,b \in \Omega$,

$$\alpha(a+b) = \alpha a+\alpha b$$

As $\alpha \in U_1$, $a,b \in \Omega$, the condition (ii) gives that $\alpha a, \alpha b \in U_1$ and the condition (i) gives that $\alpha a+\alpha b \in U_1$.

As $a,b \in \Omega$ and Ω is an m-semilattice $a+b \in \Omega$ and hence $\alpha(a+b) = \alpha a+\alpha b$ is true in U_1 In a similiar manner, other laws can be proved. Thus U_1 is a latticoid. Hence the theorem is proved.

Theorem :—*Sublatticoids of U form a lattice with $U_1 \cap U_2$ as the lattice product and U_1+U_2 as the lattice sum.*

Proof. In the Boolean algebra of subsets of U, determine a topology $X \to \bar{X}$ where X is any set and $\bar{X}$ is the smallest sublatticoid containing X. This smallest latticoid exists as the meet of two sublatticoids is a sublatticoid. The closed elements form a lattice in which the product and the sum are $U_1 \cap U_2$ and $U_1+U_2 = \overline{U_1 \cup U_2}$.

The elements of U_1+U_2 are $\alpha+\beta$ where $\alpha \in U_1$, $\beta \in U_2$

Proof. As α,β $U_1 \cup U_2$, $\alpha+\beta \in U_1+U_2 = \overline{U_1 \cup U_2}$. The set of all elements $\alpha+\beta$ where $\alpha \in U_1$, $\beta \in U_2$ form a sublatticoid.

For, $(\alpha_1+\beta_1)+(\alpha_2+\beta_2) = (\alpha_1+\alpha_2)+(\beta_1+\beta_2) = \alpha_3+\beta_3$ where $\alpha_3=\alpha_1+\alpha_2$ and $\beta_3=\beta_1+\beta_2$. As U_1,U_2 are sublatticoids and $\alpha_1,\alpha_2 \in U_1$, $\beta_1,\beta_2 \in U_2$, $\alpha_1+\alpha_2 \in U_1$, $\beta_1+\beta_2 \in U_2$. Further, as U_1,U_2 are sublatticoids and $\alpha \in U_1$, $\beta \in U_2$, $\alpha' = \alpha a \in U_1$, $\beta' = \beta a \in U_2$. Hence

$$(\alpha+\beta)a = \alpha a+\beta a = \alpha'+\beta'$$

where $\alpha' \in U_1$, $\beta' \in U_2$. Thus the set of elements $\alpha+\beta$ forms a sublatticoid. But by definition $U_1+U_2 = \overline{U_1 \cup U_2}$ is the smallest sublatticoid containing U_1 and U_2, U_1+U_2 is identical with the sublatticoid formed by the elements $\alpha+\beta$.

The representation of the elements of U_1+U_2 is not unique, unless $U_1 \cap U_2 = 0$.

For, if $U_1 \cap U_2 \neq 0$, there is an element $\gamma \in U_1 \cap U_2$. Let $\alpha \in U_1$, $\beta \in U_2$ be any two elements and put $\alpha' = \alpha+\gamma$, $\beta' = \beta+\gamma$. Then $\alpha'+\beta' = (\alpha+\gamma)+(\beta+\gamma) = \alpha+\beta+\gamma = \alpha+\beta' = \alpha'+\beta$. Thus the representation is not unique.

The sum U_1+U_2 when $U_1 \cap U_2 = 0$ is called a direct sum and is denoted as $U_1 \oplus U_2$.

3. **Dependence.** In a right latticoid U, an element α is called *dependent* on the elements

$$\alpha_1,\alpha_2,\ldots\ldots,\alpha_n$$

if $\alpha = \alpha_1 a_1+\alpha_2 a_2+\ldots\ldots+\alpha_n a_n$, n being finite. Let I be a set of indices. Let α_i $i \in I$ be a set of elements of U. An element α is said to be finitely dependent on S if α is dependent on a finite number of elements of S.

Theorem. *The element α is dependent on $\alpha_1,\alpha_2\ldots\ldots,\alpha_n$ if $\alpha a = \alpha_1 a_1+\alpha_2 a_2+\ldots\ldots+\alpha_n a_n$ and a is a reversible element of Ω.*

Proof. As a is reversible, a^{-1} exist and $aa^{-1}=1$ where 1 is the top element of Ω

Then if $\alpha a = \alpha_1 a_1+\alpha_2 a_2+\ldots\ldots+\alpha_n a_n$, one gets by multiplying with a^{-1}, $(\alpha a)a^{-1} = (\alpha a_1)a^{-1}+(\alpha_2 a_2)a^{-1}\ldots\ldots(\alpha a_n)a^{-1}$.

But $(\alpha a)a^{-1} = \alpha(a\,a^{-1}) = \alpha 1 = \alpha$ by axiom IV.

Putting $a_i a^{-1} = b_i$,

$$\alpha = \alpha_1 b_1+\alpha_2 b_2+\ldots\ldots+\alpha_n b_n \text{ where } b_i \in \Omega.$$

Hence α is dependent on $\alpha_1,\alpha_2\ldots\ldots\alpha_n$.

The elements dependent on $\alpha_1,\alpha_2,\ldots\ldots\alpha_n$ form a sublatticoid.

Proof. Let α,α' be two elements dependent on $\alpha_1,\alpha_2,\ldots\ldots,\alpha_n$.

Then
$$\alpha = \alpha_1 a_1+\alpha_2 a_2+\ldots\ldots\ldots+\alpha_n a_n$$
$$\alpha' = \alpha_1 a_1'+\alpha_2 a_2'+\ldots\ldots+\alpha_n a_n'$$

Hence $\alpha+\alpha' = \alpha_1(a_1+a_1')+\alpha_2(a_2+a_2')\ldots\ldots+\alpha_n(a_n+a_n')$. Also if a is any element of Ω.

$$\alpha a = (\alpha_1 a_1)a+(\alpha_2 a_2)a+\ldots+(\alpha_n a_n)a.$$
$$= \alpha(a_1 a)+\alpha_2(a_2 a)+\ldots+\alpha_n(a_n a).$$

Hence both $\alpha+\alpha'$ and αa where $a \in \Omega$ are dependent on $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus the proposition hold.

If α is not dependent on $\alpha_1, \alpha_2, \ldots, \alpha_n$, then α is called independent of $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus if α is independent of $\alpha_1, \alpha_2, \ldots \alpha_n$,

$$\alpha \neq \alpha_1 a_1+\alpha_2 a_2+\ldots+\alpha_n a_n, \text{ for all } a_1, a_2, \ldots, a_n.$$

The set of independent elements of a latticoid is called a basis of the latticoid.

The smallest latticoid containing a given set S of elements of Ω is called a latticoid generated by S.

Theorem. *Let S be a subset of a latticoid U. In order that an element $\alpha \in U$ may belong to the sublatticoid V, generated by S, it is necessary and sufficient that α is finitely dependent on S.*

Proof. Let α be dependent on the subset S of elements.

Then, by the definition of dependence,

$$\alpha = \alpha_1 a_1+\alpha_2 a_2+\ldots\ldots+\alpha_n a_n,\ \alpha_i \in S.$$

As $\alpha_i \in S$, the set A_i of all $\alpha_i a$ form a sublatticoid of U.

For $\quad (\alpha_i a)b = \alpha_i(ab) \in A_i \quad$ and $\quad \alpha_i a+\alpha_i b = \alpha_i(a+b) \in A_i$.

The set, $\quad \alpha_1 a_1+\alpha_2 a_2+\ldots+\alpha_n a_n \in A_1+A_2+\ldots+A_n \in V.$

Hence $\quad \alpha \in V.$

Thus the condition is sufficient.

The condition is also necessery.

For, let V be generated by S. Then if $\alpha_1, \alpha_2, \ldots \alpha_n$ are elements of $S, \ldots \alpha_1 a_1+\alpha_2 a_2+\ldots\ldots+\alpha_n a_n \in V$. These elements form a sublatticoid of V. As by difinition, V is the smallest sublatticoid generated by S, the elements of V are expressible as

$$\alpha_1 a_1+\alpha_2 a_2+\ldots+\alpha_n a_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a finite number of elements of S. Hence the elements of V are finitely dependent on S.

4. Isotone mappings. A homomorphism or a linear mapping of a latticoid U into another latticoid V, both of which are over the same m-semilattice Ω, is a homomorphism of the upper semilattice U into the upper semilattice V such that

$f(\alpha a) = f(\alpha)a$ for any $\alpha \in U$ $a \in \Omega$. Thus the linear mapping is such that

$$f(\alpha+\beta) = f(\alpha)+f(\beta)$$
$$f(\alpha a) = f(\alpha)a.$$

This is a special case of the isotone mapping of the latticoid. An *isotone mapping* f of the latticoid U into the latticoid V is a mapping f which keeps the inclusion invariant and is such that $f(\alpha a) = f(\alpha)a$. Similarly one can define antitone mapping of the latticoid. The set of isotone mappings of a latticoid U into another latticoid V will be denoted as Iso (U, V). The set of all linear mappings $U \to V$ will be denoted as Hom (U, V). Then Hom $(U, V) \subseteq$ Iso (U, V)

Theorem. *Iso (U, V) of two right latticoids U, V, is an upper semilattice. Hom (U, V) is a sub-semilattice of Iso (U, V).*

Proof. Define the sum of two isotone mappings f, g, by the rule

$$(f+g)\alpha = f(\alpha)+g(\alpha) \text{ where } \alpha \in U.$$

The sum $f+g$ so defined is isotone; for, let α, β be two elements of U such that $\alpha \subseteq \beta$.

Then
$$(f+g)\alpha = f\alpha+g\alpha$$
$$(f+g)\beta = f\beta+g\beta$$

As f, g, are isotone, $f(\alpha) \subseteq f(\beta)$ and $g(\alpha) \subseteq g(\beta)$ Hence $f(\alpha)+g(\alpha) \subseteq f(\beta)+g(\beta)$.

$$\therefore (f+g)\alpha \subseteq (f+g)\beta.$$

Also
$$(f+g)(\alpha a) = f(\alpha a)+g(\alpha a) = f(\alpha)a+g(\alpha)a = [f(\alpha)+g(\alpha)]a$$
$$= [(f+g)\alpha]a.$$

Hence
$$f+g \in \text{Iso } (U, V).$$

It follows from the definition of the sum, that

$$f+g = g+f, \ (f+g)+h = f+(g+h) \qquad f+f = f.$$

Thus Iso (U, V) is an upper semilattice. As linear maps are isotone, Hom (U, V) is a subsemilattice of Iso (U, V).

The mapping I, defined by the condition that

$$1\alpha = 1 \in V \quad \text{for all } \alpha \in U$$

is the topmost element of Iso (U, V).

For,
$$(1+h)\alpha = 1\alpha+h\alpha = 1+h\alpha = 1 = 1\alpha$$

Hence
$$1+h = 1 \text{ i.e. } h \subseteq 1$$

Theorem. *Iso (U, V) is an upper m-semilattice with the identity mapping i as the identity.*

Proof. In the above theorem, Iso (U, V) has been proved to be an upper semilattice.

Now define the product fg by the rule

$$(fg)\alpha = f(g\alpha).$$

This product is associative,

For
$$[(fg)h]\alpha = f(g(h\alpha)) = [f(gh)]\alpha$$

It is also right distributive.

For
$$[(f+g)h]\alpha = (f+g)(h\alpha)$$
$$= (f+g)\alpha' \quad \text{when } h\alpha = \alpha' \in U$$
$$= f\alpha' + g\alpha'$$
$$= fh\alpha + gh\alpha = (fh+gh)\alpha$$

If i is the identity mapping,
$$i\alpha = \alpha$$

Then $gi\alpha = g\alpha$. Hence $gi = g$.

Similarly $ig = g$.

A two sided latticoid is one which is both a left latticoid as well as a right latticoid and in which $(a\alpha)b = a(\alpha b)$ hold.

Theorem. *If U is a right latticoid and V is a two sided latticoid over an upper m-semilattice Ω, then Iso (U, V) can be made a left latticoid.*

Proof. As V is a right latticoid, Iso (U, V) is an upper semilattice.

Now define af, $a \in \Omega$, $f \in$ Iso (U, V) by the condition that $(af)\alpha = a(f\alpha)$

This is possible, as $f\alpha \in V$ which is a two sided latticoid.

Then $af \in$ Iso (U, V).

For $(af)(\alpha b) = a[f(\alpha b)] = a[(f\alpha)b]$ (by the rule $a(rb) = (ar)b$ in V) $= [(af)\alpha]b$

Also let $\alpha \sqsubseteq \beta$, $\alpha, \beta \in U$. Then $f(\alpha) \sqsubseteq f(\beta)$ in V as f is isotone.

So $af(\alpha) \sqsubseteq af(\beta)$, since the left scalar multiplication is isotone in V.

Thus $(af)\alpha \sqsubseteq (af)\beta$.

Hence $af \in$ Iso (U, V).

It is easy to verify that
$$a(f+g) = af+ag$$
$$(a+b)f = af+bf$$
$$if = f = fi.$$

Thus in Iso (U, V), one can define a left scalar multiplication making it a left latticoid.

In the above theorem, Ω is a right distributive m-semilattice. Suppose that Ω is both sided distributive. In this case, Ω is a two sided latticoid provided the product is considered as the scalar products. For,

(i) $ab \in \Omega$, when $a, b \in \Omega$

(ii) $a(b+c) = ab+ac$, $(b+c)a = ba+ca$

(iii) $(ab)c = a(ac)$

(iv) $a1 = a$

Iso (U, Ω) is a left latticoid which will be denoted as U^* i.e. $U^* =$ Iso (U, Ω). U^* will be called the dual of U.

If U is a right latticoid, then U^* has been proved to be a left latticoid. In a similar manner, it can be proved that if U is a left latticoid, then U^* is a right latticoid. then U^* is a right latticoid. Consequently if U is a right latticoid, $U^{**} = (U^*)^*$ is a right latticoid U^{**} will be called the bidual of U.

We shall now study the relation between U and U^{**}. Let α be a fixed element of U and $f \in U^* = \mathrm{Iso}(U, \Omega)$. Then $f(\alpha) \in \Omega$. Denote by φ_α; the mapping

$$f \rightarrow f(\alpha)$$

maps U^* into Ω. Then $\varphi_\alpha(f) = f(\alpha)$.

Theorem. *The mapping* $\varphi_\alpha : U^* \rightarrow \Omega$ *is isotone on the left latticoid* U^*.

Proof. $\varphi_\alpha(af) = (af)\alpha = a(f\alpha) = a\varphi_\alpha(f)$.

Also let $f \subseteq g$. Then $f + g = g$.

Hence $\varphi_\alpha(f+g) = \varphi_\alpha(g)$

But $\varphi_\alpha(f+g) = (f+g)\alpha = f\alpha + g\alpha = \varphi_\alpha(f) + \varphi_\alpha(g)$.

Thus $\varphi_\alpha(g) = \Phi_\alpha(f) + \varphi_\alpha(g)$

Hence $\varphi_\alpha(f) \subseteq \varphi_\alpha(g)$

and φ_α is isotone on U^*.

Writing $f(\alpha)$ as $f\alpha$, one notes that $\varphi_\alpha(f) = f\alpha$ and

hence $f \subseteq g$ implies $f\alpha \subseteq g\alpha$.

Also $\alpha \subseteq \beta$ implies $f\alpha \subseteq f\beta$.

Thus one gets that the product $f\alpha$ is isotone with each factor.

As φ_α is isotone on U^*, φ_α is an element of U^{**}. The mapping: $\alpha \rightarrow \varphi_\alpha$ is a natural mapping of U into U^{**}.

Theorem. *The natural mapping* $\psi : U \rightarrow U^{**}$ *is isotone.*

Proof. Let α, β be two elements of U such that $\alpha \subseteq \beta$. Suppose that f is an isotone mapping of U, so that $f \in U$.

Then $f(\alpha) \subseteq f(\beta)$.

As $\varphi_\alpha(f) = f(\alpha)$, $\varphi_\beta(f) = f(\beta)$,

$$\varphi_\alpha(f) \subseteq \varphi_\beta(f).$$

As this is true for every $f \in U^*$, $\varphi_\alpha \subseteq \varphi_\beta$.

As $\psi(\alpha) = \varphi_\alpha$, this gives that $\psi(\alpha) \subseteq \psi(\beta)$.

Also $\varphi_{\alpha a}(f) = f(\alpha a) = f(\alpha)a = \varphi_\alpha(f)a = [\varphi_\alpha a]f$

Hence $\varphi_{\alpha a} = \varphi_\alpha a$ *i.e.* $\psi(\alpha a) = \psi(\alpha)a$.

Thus the natural mapping ψ is isotone.

Next let U, V be two latticoids over Ω and $f: U \to V$ be an isotone mapping *i.e.* $f \in \mathrm{Iso}(U, V)$.

If $v \in V^*$, v maps V into Ω and hence $vf: U \xrightarrow{f} V \xrightarrow{v} \Omega$ is an isotone mapping of U into Ω and is an element of U^*. Then the map

$$f^t . v \to vf$$

is a map $V^* \to U^*$ and will be denoted as f^t. It will be called the transpose of f.

Theorem. *The transpose f^t is isotone* i.e. $f^t \in \mathrm{Iso}\,(V^*, U^*)$

Proof. Let $v_1 \subseteq v_2$ where $v_1, v_2 \in V^*$.

then as v_1, v_2, f are all isotone, $v_1 f \subseteq v_2 f$.

But as $f^t . v \to vf \quad f^t(v) = vf$.

Hence $f^t(v_1) \subseteq f^t(v_2)$.

Further, as $a \in \Omega$, $x \in U$, then

$$[(va)f]x = (va)(fx) = (v(fx))a) = (vf)x\, a$$
$$= (vf\,a)x.$$

Hence $f^t(va) = f^t(v)a$.

Thus f^t is isotone and $f^t \in \mathrm{Iso}(V^*, U^*)$.

The mapping $\psi: f \to f^t$ is a mapping of $\mathrm{Iso}\,(U, V)$ into $\mathrm{Iso}\,(V^*, U^*)$.

Theorem. *The mapping ψ is isotone* i.e. $\psi \in Iso(Iso(U, V), Iso(V, U))$

Proof. As $\psi: f \to f^t$, $\psi(f) = f^t$.

Now let $f \subseteq g$. Then $vf \subseteq vg$. *i.e.* $f^t(v) \subseteq g^t(v)$ for all $v \in V^*$.

Thus $f^t \subseteq g^t$ *i.e.* $\psi(f) \subseteq \psi(g)$

This proves that ψ is isotone.

Now let $a \in \Omega$, $v \in V^*$.

Then $(af^t)v = v(af)$.

If $x \in U$, $(v(af))x = v(af(x)) = a\, vf(x)) = a((vf)x)$.

Hence $v(af) = a(vf)$.

Thus $(af^t)v = af^t(v)$ for all $v \in V^*$.

Hence $\psi(af) = a\psi(f)$ *i.e.* ψ admits the scalar multiplication.

Hence ψ is isotone.

This imply that $\psi \in \mathrm{Iso}(\mathrm{Iso}(U, V), \mathrm{Iso}(U^*, V^*)$.

One can evidently generalise this result.

Let $U \xrightarrow{f} V \xrightarrow{g} W$ be two isotone maps. Then $f^0 g \to gf$ is a map $\mathrm{Iso}\,V, W) \to \mathrm{Iso}(U, W$

This map f^0 is isotone *i.e.* $f^0 \in \mathrm{Iso}(\mathrm{Iso}(V, W), \mathrm{Iso}(U, W))$.

Further the map $\psi: f \to f^0$

is isotone and hence $\psi \in \text{Iso}\,(\text{Iso}\,(U,V),\ \text{Iso}\,(\text{Iso}\,(V,W),\ \text{Iso}\,(U,W)))$.

Theorem. *If $U \xrightarrow{f} V \xrightarrow{g} W$ are isotone, then*

$$(gf)^t = f^t g^t.$$

Proof. Let $w \in W^*$ *i.e.* let w be an isotone map $W \to \Omega$.

As
$$U \xrightarrow{f} V \xrightarrow{g} W$$

$$gw = v \text{ is an isotone map } V \to \Omega.$$

Also by definition,

$$f^t : v \to vf$$

$$g^t : w \to wg$$

$$(gf)^t : w \to w(gf)$$

Now $(gf)^t w = w(gf)$

$= (wg)f$, as w g,f are mappings and hence the product is associative

$= g^t(w)f$.

As
$$g^t(w) = wg = v,$$

so
$$(gf)^t w = vf = f^t(v)$$
$$= f^t(g^t(w))$$
$$= f^t g^t(w).$$

As this is true for every w,

$$(gf)^t = f^t g^t.$$

All the above considerations have been made for the isotone mappings which form $\text{Iso}(U,V)$. But these can also be made for linear mappings $U \to V$.

5. **Representation.** Let R be a given upper m semilattice and V be a given upper semilattice.

A homomorphism $f : R \to E(U)$ where $E(U)$ is the upper m-semilattice of isotone mappings of U, is called a *representation* of R.

Theorem. *A representation f of R determines a left latticoid over R and conversely.*

Proof. Let f be a given mapping $R \to E(U)$. Then by $f, r \to a$ where a is an element of $E(U)$. Now define a scalar multiplication $r\alpha$ by the rule $r\alpha = a(\alpha) = f(r)\alpha$ where $r \in R$ and $\alpha \in U$. As $a \in E(U)$, a is an endomapping of U and consequently $a(\alpha) \in U$. Thus $r\alpha$ is a scalar multiplication of the elements of U with those of R.

As a is an endomorphism of the upper semilattice U,

$$a(\alpha_1+\alpha_2) = a(\alpha_1)+a(\alpha_2)$$

Hence $$r(\alpha_1+\alpha_2) = r\alpha_1+r\alpha_2$$

Also, by the definition of the sum of two endomorphisms

$$(a_1+a_2)\alpha = a_1\alpha+a_2\alpha$$

Hence $$(r_1+r_2)\alpha = r_1\alpha+r_2\alpha$$

Thus the scalar product satisfies both distributive laws.

Also, by the definition of the product of two endomorphisms

$$(a_1a_2)\alpha = a_1(a_2\alpha)$$

Hence $$(r_1r_2)\alpha = r_1(r_2\alpha)$$

Consequently U becomes a latticoid over R.

Conversely, let U be a latticoid over an upper m-semilattice R. Then if r is a given element of R, define a mapping $a: U\to U$ by the rule

$$a: \alpha\to r\alpha, \text{ where } \alpha \in U$$

Then $$a(\alpha) = r\alpha.$$

As $$r(\alpha_1+\alpha_2) = r_1\alpha+r_2\alpha,\ a(\alpha_1+\alpha_2) = a(\alpha_1)+a(\alpha_2).$$

Thus a is an endomorphism of U.

Also as $$(r_1+r_2)\alpha = r_1\alpha+r_2\alpha,\ (a_1+a_2)(\alpha) = a_1(\alpha)+a_2(\alpha).$$

Consequently the sum of the mappings a_1, a_2 is given by this law. Lastly, as $(r_1r_2)\alpha = r_1(r_2\alpha)$, $(a_1a_2)\alpha = a_1(a_2(\alpha))$. This gives the product of two mappings.

With these definitions of sum and product, these endomorphisms form an upper m semilattice $E(U)$ The mapping $f: R\to E(U)$ is a homomorphism. For, let $f: r\to a \in E(U)$. Then $f(r) = a$ and $f(r)\alpha = a(\alpha) = r\alpha$ where $\alpha \in U$. Consequently $f(r_1+r_2)\alpha = (r_1+r_2)\alpha = r_1\alpha+r_2\alpha = f(r_1)\alpha+f(r_2)\alpha = (f(r_1)\alpha+f(r_2)\alpha$.

$$f(r_1r_2)\alpha = (r_1r_2)\alpha = r_1(r_2\alpha) = f(r_1)(f(r_2)\alpha) = (f(r_1)f(r_2))\alpha.$$

Thus $$f(r_1+r_2) = f(r_1)+f(r_2),\ f(r_1r_2) = f(r_1)f(r_2)$$

Consequently f is a homomorphism of the upper m-semilattice R into the upper m-semilattice $E(U)$. Hence f is a representation of R through U.

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NOTE ON THE STRESSES DUE TO A CENTRE OF DILATATION IN AN INFINITE SLAB WITH FIXED PLANE FACES

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Introduction. The effect of a centre of dilatation in a semi-infinite elastic solid with its plane face *fixed* has been determined by the author (1960) in a previous paper. The object of the present note is to find the stresses and displacements due to a centre of dilatation inside an infinite, isotropic elastic slab of finite and uniform thickness when parallel plane faces of the slab are kept fixed.

Solution. We take one of the plane faces of the slab as the plane $z=0$ with the origin on this face. The axis of z is drawn into the body. If there be a centre of dilatation at the point $(0, 0, c)$, we have the displacements given by (*cf.* Bhowmick, 1960)

$$\left.\begin{aligned} u &= Px\left[-\frac{1}{R_1^3}+\frac{1}{R_2^3}-\frac{6z(z+c)}{(3-4\sigma)R_2^5}\right],\\ v &= Py\left[-\frac{1}{R_1^3}+\frac{1}{R_2^3}-\frac{6z(z+c)}{(3-4\sigma)R_2^5}\right],\\ w &= P\left[-\frac{z-c}{R_1^3}-\frac{z+c}{R_2^3}+\frac{2z}{(3-4\sigma)R_2^3}-\frac{6z(z+c)^2}{(3-4\sigma)R_2^5}\right]. \end{aligned}\right\} \tag{1}$$

where
$$R_1^2 = x^2+y^2+(z-c)^2,$$
$$R_2^2 = x^2+y^2+(z+c)^2,$$

P is a constant depending on the strength of the nucleus, and σ is Poisson's ratio.

It is evident that $u=v=w=0$ when $z=0$. Writing r for $(x^2+y^2)^{\frac{1}{2}}$,

we have
$$\left.\begin{aligned} \frac{1}{R_1} &= \int_0^\infty e^{-\alpha(z-c)}J_0(\alpha r)\,d\alpha.\ z>c\\ \frac{1}{R_2} &= \int_0^\infty e^{-\alpha(z+c)}J_0(\alpha r)\,d\alpha, \end{aligned}\right\} \tag{2}$$

$J_0(\alpha r)$ being a Bessel's function of the first kind and zero order. We can write u, v, w as

$$u = P\left\{\int_0^\infty e^{-\alpha(z-c)}\frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha - \int_0^\infty e^{-\alpha(z+c)}\frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha - 2\int_0^\infty \alpha z e^{-\alpha(z+c)}\frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha\right\},$$

$$v = P\left\{\int_0^\infty e^{-\alpha(z-c)}\frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha - \int_0^\infty e^{-\alpha(z+c)}\frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha - 2\int_0^\infty \alpha z e^{-\alpha(z+c)}\frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha\right\},$$

$$w = P\left\{\int_0^\infty -\alpha e^{-\alpha(z-c)}J_0(\alpha r)d\alpha + \int_0^\infty -\alpha e^{-\alpha(z+c)}J_0(\alpha r)d\alpha - \frac{2}{3-4\sigma}\int_0^\infty \alpha^2 z e^{-\alpha(z+c)}J_0(\alpha r)d\alpha\right\}.$$

If we ta the other face of the slab as given by $z = 2c$, we have on this face

$$\left.\begin{aligned} [u]_{z=2c} &= \int_0^\infty Q(\alpha)\frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha, \\ [v]_{z=2c} &= \int_0^\infty Q(\alpha)\frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha, \\ [w]_{z=2c} &= \int_0^\infty R(\alpha)J_0(\alpha r)d\alpha, \end{aligned}\right\} \tag{3}$$

where
$$Q(\alpha) = P(e^{-\alpha c} - 3e^{-3\alpha c} - 4\alpha c e^{-3\alpha c}),$$
$$R(\alpha) = -P[\alpha e^{-\alpha c} + \alpha e^{-3\alpha c} + 4\alpha^2 c(3-4\sigma)^{-1}e^{-3\alpha c}].$$

We are to find another system of displacements (U, V, W) which satisfy the equations of equilibrium, vanish at $z = 0$, and nullify the displacements (3) on $z = 2c$.

In the absence of body forces, equations of equilibrium are (*cf.* Love, 1927, p. 133)

$$\nabla^2(U, V, W) = -\frac{1}{1-2\sigma}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Delta.$$

Let
$$\Delta = -2(1-2\sigma)\frac{\partial F}{\partial z}$$
such that
$$\nabla^2 F = 0,$$
and
$$\nabla^2(U, V, W) = \nabla^2\left(z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\right)F.$$

Hence we can take
$$U = z\frac{\partial F}{\partial x} + \varphi_1,$$
$$V = z\frac{\partial F}{\partial y} + \varphi_2,$$
$$W = z\frac{\partial F}{\partial z} + \varphi_3,$$

where $\varphi_1, \varphi_2, \varphi_3$ are harmonic functions,

Let
$$\varphi_1 = \int_0^\infty A(\alpha) \sinh \alpha z \frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha,$$
$$\varphi_2 = \int_0^\infty A(\alpha) \sinh \alpha z \frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha,$$
$$\varphi_3 = \int_0^\infty \alpha B(\alpha) \sinh \alpha z\, J_0(\alpha r) d\alpha.$$

Since
$$\Delta = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = -2(1-\sigma)\frac{\partial F}{\partial z}$$

we shall have
$$\frac{\partial F}{\partial z} = \frac{1}{4\sigma-3}\int_0^\infty \alpha^2(B \cosh \alpha z - A \sinh \alpha z)J_0(\alpha r)d\alpha$$

whence
$$F = \frac{1}{4\sigma-3}\int_0^\infty \alpha(B \sinh \alpha z - A \cosh \alpha z)J_0(\alpha r)d\alpha.$$

The above value of F gives

$$U = \frac{1}{4\sigma-3}\int_0^\infty \alpha z\,(B \sinh \alpha z - A \cosh \alpha z)\frac{\partial}{\partial x}[J_0(\alpha z)]d\alpha + \int_0^\infty A \sinh \alpha z \frac{\partial}{\partial x}[J_0(\alpha r)]d\alpha,$$

$$V = \frac{1}{4\sigma-3}\int_0^\infty \alpha z(B \sinh \alpha z - A \cosh \alpha z)\frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha + \int_0^\infty A \sinh \alpha z \frac{\partial}{\partial y}[J_0(\alpha r)]d\alpha,$$

$$W = \frac{1}{4\sigma-3}\int_0^\infty \alpha^2 z\,(B \cosh \alpha z - A \sinh \alpha z)J_0(\alpha r)d\alpha + \int_0^\infty \alpha B \sinh \alpha z\, J_0(\alpha r)d\alpha.$$

On $z = 2c$ we must have $u+U = 0,\ v+V = 0,\ w+W = 0.$

Therefore
$$\frac{2c\alpha}{4\sigma-3}(B \sinh 2\alpha c - A \cosh 2\alpha c) + A \sinh 2\alpha c = -Q(\alpha)$$

$$\frac{2c\alpha^2}{4\sigma-3}(B \cosh 2\alpha c - A \sinh 2\alpha c) + \alpha B \sinh 2\alpha c = -R(\alpha).$$

From the above equations we get

$$A(\alpha) = -\frac{(3-4\sigma)[R(\alpha)2c \sinh 2\alpha c + Q(\alpha)\{(3-4\sigma)\sinh 2\alpha c - 2\alpha c \cosh 2\alpha c\}]}{[(3-4\sigma)^2 \sinh^2 2\alpha c - 4\alpha^2 c^2]}$$

$$B(\alpha) = \frac{(3-4\sigma)[Q(\alpha)2c\alpha \sinh 2\alpha c - R(\alpha)\alpha^{-1}\{(3-4\sigma)\sinh 2\alpha c + 2\alpha c \cosh 2\alpha c\}]}{[(3-4\sigma)^2 \sinh^2 2\alpha c - 4\alpha^2 c^2]}.$$

Values of stress components can be calculated from the components of displacement $u+U$, $v+V$ and $w+W$. If $\widehat{zz}_1$ and $\widehat{zz}_2$ be the normal stresses deduced from (u,v,w) and (U,V,W) respectively, we obtain

$$[\widehat{zz}_1]_{r=0} = \frac{2EP}{(1+\sigma)(3-4\sigma)}\left[\frac{3c^2}{R^5} - \frac{1}{R^3}\right]$$

where $R^2 = x^2 + y^2 + c^2$,

$$[\widehat{zz}_2]_{z=0} = + \frac{E}{(1+\sigma)(3-4\sigma)} \int_0^\infty \alpha^2 B(\alpha) J_0(\alpha r)\, d\alpha.$$

Putting $\sigma = .25$ and $c = 1$, it is found that at $r = 0$ on the face $z = 0$.

$$[\widehat{zz}]_{\substack{z=0\\ r=0}} = [\widehat{zz}_1 + \widehat{zz}_2]_{\substack{z=0\\ r=0}} = 2.28EP \text{ approximately.}$$

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SOME FORMULAE CONNECTING SELF-RECIPROCAL FUNCTIONS

By

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1. B. Mohan (1934) gave certain theorems connecting different classes of functions self-reciprocal in the Hankel transform. In this note an attempt is being made to generalise those theorems. In the end a few new results are obtained therefrom.

Following Hardy and Titchmarsh we shall say that a function is R_ν, if it is self-reciprocal for J_ν transforms and it is $-R_\nu$, if it is skew-reciprocal for J_ν transforms. Also, for $R_{\frac{1}{2}}$ and $R_{-\frac{1}{2}}$ we shall write R_s and R_c respectively.

We shall make use of the following known result (Hardy and Titchmarsh, 1930).

A necessary and sufficient condition that a function $f(x)$ should be R_μ is that it should be of the form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\psi(s)x^{-s}ds, \tag{1.1}$$

where $\quad 0 < c < 1, \qquad$ *and* $\qquad \psi(s) = \psi(1-s). \qquad (1.2)$

2. One of the theorems of B. Mohan is

If $f(x)$ is R_μ, the function

$$g(x) = \frac{1}{x}\int_0^x Q\left(\log\frac{x}{y}\right)f(y)dy$$

is R_ν, provided that

$$\left.\begin{aligned} Q(x) &= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\omega(s)e^{xs}ds \qquad & (x>0)\\ &= 0, & (x<0)\end{aligned}\right\} \tag{2.1}$$

where k is any positive integer and $\omega(s)$ satisfies (1.2)

We generalise this theorem in the following form.

Theorem I. *If $f(x)$ is of the form*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\chi(s)x^{-s}ds \tag{2.2}$$

where $\chi(s)$ satisfies (1.2),

then the function $$g(x) = \frac{1}{x}\int_0^x Q\left(\log\frac{x}{y}\right)f(y)dy$$

is R_λ, *provided that* $Q(x)$ *is given by* (2.1).

By (2.2) we have

$$g(x) = \frac{1}{2\pi i x}\int_0^x Q\left(\log\frac{x}{y}\right)dy\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\chi(s)y^{-s}ds,$$

where $\lambda(s) = \chi(1-s)$. Hence, provided the inversion of the order of integration is justified, we have

$$g(x) = \frac{1}{2\pi i x}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\chi(s)ds\int_0^x Q\left(\log\frac{x}{y}\right)y^{-s}dy,$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\chi(s)x^{-s}ds\int_0^x Q(u)e^{(s-1)u}du.$$

Now, using a form of Mellin's Inversion Formula (Hardy, 1921) from (2.1) we have

$$\int_0^\infty Q(x)e^{-sx}dx = \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\omega(s),$$

where $\omega(s) = \omega(1-s)$. Changing s into $1-s$, we get

$$\int_0^\infty e^{(s-1)x}Q(x)dx = \Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\omega(s).$$

Hence

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)$$
$$\times\,\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu-\tfrac{1}{2}s)\chi(s)\omega(s)x^{-s}ds,$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\lambda+\tfrac{1}{2}s)\psi(s)x^{-s}ds,$$

where $$\psi(s) = \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\nu-\tfrac{1}{2}s)\chi(s)\omega(s)$$

As $\psi(s)$ satisfies the equation $\psi(s) = \psi(1-s)$, it follows from (1.1) that $g(x)$ is R_λ.

3. **Corollary 1**. *When* $\lambda = \nu$, *the above theorem reduces to the proposition established by B. Mohan referred to above.*

Corollary 2. *When* $\mu = \nu$, *the above corollary reduces further to the following simpler form:*

If $f(x)$ *is* R_ν, *the function*

$$g(x) = \frac{1}{x}\int_0^x Q\left(\log\frac{x}{y}\right)f(y)dy$$

is R_ν, *provided that*

$$Q(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} e^{xs}\lambda(s)ds \qquad (x>0)$$
$$= 0, \qquad (x<0)$$

where $$\lambda(s) = \lambda(1-s).$$

Here we shall give a few interesting examples of this theorem.

(*i*) Let

$$\lambda(s) = \sum_{m=0}^{\infty}\left[\frac{(-1)^m(\frac{1}{2}b)^{\gamma+2m-1}(\frac{1}{2}a)^{\alpha-\beta}\Gamma(2\alpha+2m)}{m!\,\Gamma(\gamma+m)\Gamma(\alpha-\beta+1)\{a^2+(s-\frac{1}{2})^2\}^{\alpha+m}}\, {}_2F_1\left\{\begin{matrix}\alpha+m, \frac{1}{2}-\beta-m\\ \alpha-\beta+1\end{matrix}; \frac{a^2}{a^2+(s-\frac{1}{2})^2}\right\}\right]$$

where $R(b)>0, |I(b)|<(s-\frac{1}{2})$, and $|b| \lessgtr \sqrt{[a^2+(s-\frac{1}{2})^2]}-(s-\frac{1}{2})$.

Then, by a formula given by Watson (1944), we get

$$Q(x) = e^{\frac{1}{2}x}\frac{J_{\alpha-\beta}(ax)J_{\gamma-1}(bx)}{x^{\gamma-\alpha-\beta}} \qquad (x>0)$$
$$= 0, \qquad (x \lessgtr 0).$$

Hence $$g(x) = \frac{1}{x^{\frac{1}{2}}}\int_0^x \frac{J_{\alpha-\beta}\left(a\log\frac{x}{y}\right)J_{\gamma-1}\left(b\log\frac{x}{y}\right)}{y^{\frac{1}{2}}\left(\log\frac{x}{y}\right)^{\lambda-\alpha-\beta}}f(y)dy.$$

As a particular case, when $\lambda = \alpha+\beta$ we get

$$g(x) = \frac{1}{x^{\frac{1}{2}}}\int_0^x \frac{1}{y^{\frac{1}{2}}}J_{\alpha-\beta}\left(a\log\frac{x}{y}\right)J_{\alpha+\beta-1}\left(b\log\frac{x}{y}\right)f(y)dy.$$

(*ii*) Let $$\lambda(s) = \frac{1}{\pi(bc)^{\frac{1}{2}}}J^{1}_{n-\frac{1}{2}}\left[\frac{(s-\frac{1}{2})^2+b^2+c^2}{2bc}\right],$$

where $R(s) > (\frac{1}{2}\pm ib\pm ic)$ and $R(n)>-\frac{1}{2}$. Then, with the help of another formula given by Watson (1944), we get

$$Q(x) = e^{\frac{1}{2}x}J_n(bx)J_n(cx) \qquad (x>0)$$
$$= 0, \qquad (x<0)$$

Hence $$g(x) = \frac{1}{x^{\frac{1}{2}}}\int_0^x \frac{1}{y^{\frac{1}{2}}}J_n\left(b\log\frac{x}{y}\right)J_n\left(c\log\frac{x}{y}\right)f(y)dy.$$

(*iii*) Let

$$\lambda(s) = \frac{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\lambda+\frac{1}{2})\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\lambda+\frac{1}{2})\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\lambda)\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\lambda)}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})a^{-\lambda}2^{2-\alpha}(s-\frac{1}{2})^{\alpha+\lambda}}\, {}_2F_1\left\{\begin{matrix}\frac{1}{2}\alpha+\frac{1}{2}\lambda+\frac{1}{2}, \frac{1}{2}\alpha+\frac{1}{2}\lambda\\ \alpha+\frac{1}{2}\end{matrix}; \frac{(s-\frac{1}{2})^2-a^2}{(s-\frac{1}{2})^2}\right\}$$

where $a \leq s-\frac{1}{2}$ and $\alpha+\lambda$ is an even integer. Then, by a formula given by Titchmarsh (1927), we get

$$\left.\begin{aligned} Q(x) &= x^{\alpha-1}e^{\frac{1}{2}x}K_{\lambda}(ax) && (x>0) \\ &= 0, && (x<0) \end{aligned}\right\},$$

where $K_\nu(x)$ is a Bessel's function of imaginary argument of the second kind.

Hence $$g(x) = \frac{1}{x^{\frac{1}{2}}}\int_0^x \frac{1}{y^{\frac{1}{2}}}\left(\log\frac{x}{y}\right)^{\alpha-1} K_\lambda\left(a\log\frac{x}{y}\right)f(y)dy.$$

Putting $\alpha=1$ we get, as a particular case,

$$g(x) = \frac{1}{x^{\frac{1}{2}}}\int_0^x \frac{1}{y^{\frac{1}{2}}} K_\lambda\left(a\log\frac{x}{y}\right)f(y)dy,$$

where λ is an odd integer.

(*iv*) Let

$$\lambda(s) = \frac{2^{p-\frac{1}{2}}b^{n+1}\Gamma(\frac{3}{4}+\frac{1}{2}n+\frac{1}{2}p)\Gamma(\frac{1}{4}+\frac{1}{2}n+\frac{1}{2}p)}{\pi\Gamma(n+\frac{1}{2})(s-\frac{1}{2})^{n+p+\frac{1}{2}}}\,{}_3F_2\left[\begin{matrix}1,\ \frac{3}{4}+\frac{1}{2}n+\frac{1}{2}p,\ \frac{1}{4}+\frac{1}{2}n+\frac{1}{2}p \\ \frac{3}{2},\ n+\frac{3}{2}\end{matrix};\frac{-b^2}{(s-\frac{1}{2})^2}\right],$$

where b and $s-\frac{1}{2}$ are both positive and $n+p+\frac{1}{2}$ is an even positive integer.

Then, by a formula given by B. Mohan (1941), we get

$$\left.\begin{aligned} Q(x) &= x^{p-3/2}e^{\frac{1}{2}x}\boldsymbol{H}_n(bx) && (x>0) \\ &= 0, && (x<0) \end{aligned}\right\},$$

where $\boldsymbol{H}_\nu(x)$ is a Struve's function of order ν.

Hence $$g(x) = \frac{1}{x^{\frac{1}{2}}}\int_0^x \frac{1}{y^{\frac{1}{2}}}\left(\log\frac{x}{y}\right)^{p-3/2}\boldsymbol{H}_n\left(b\log\frac{x}{y}\right)f(y)dy.$$

Putting $p=\frac{3}{2}$ as a particular case we get

$$g(x) = \frac{1}{x^{\frac{1}{2}}}\int_0^x \boldsymbol{H}_n\left(b\log\frac{x}{y}\right)f(y)dy,$$

where n is zero or an even positive integer.

(*v*) Let $$\lambda(s) = \frac{2}{\pi^{\frac{1}{2}}}\frac{\Gamma(\varrho)\Gamma(\varrho+\frac{1}{2})}{(s-\frac{1}{2})^{2\varrho}}2^{n-2}\,{}_{p+2}F_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\ \varrho,\ \varrho+\frac{1}{2} \\ \beta_1,\ldots,\beta_q\end{matrix};\frac{-4b^2}{(s-\frac{1}{2})^2}\right],$$

where ϱ is a positive integer

(a) for all $s-\frac{1}{2}$ and b when $p+2\leq q$

(b) for $|b| < \frac{|s-\frac{1}{2}|}{2}$ when $p+1=q$

Then, by a formula given by Sinha (1944), we get

$$\left.\begin{aligned} Q(x) &= x^{2\rho-1}e^{\frac{1}{2}x}\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots\beta_q\end{matrix}; -b^2x^2\right] && (x>0)\\ &= 0, && (x<0)\end{aligned}\right\}.$$

Hence $g(x) = \dfrac{1}{x^{\frac{1}{2}}}\displaystyle\int_0^x \left(\log\frac{x}{y}\right)^{2\rho-1}\frac{1}{y^{\frac{1}{2}}}\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots\beta_q\end{matrix}; -\left(b\log\frac{x}{y}\right)^2\right]f(y)dy.$

4. One more theorem due to B. Mohan states that:

If $f(x)$ is R_μ, the function

$$g(x) = \int_0^{1/x} Q\left(\log\frac{1}{xy}\right)f(y)dy,$$

is R_ν, provided that

$$\left.\begin{aligned} Q(x) &= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^s\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\varrho(s)e^{xs}ds && (x>0)\\ &= 0, && (x<0)\end{aligned}\right\} \tag{4.1}$$

where $\rho(s) = \varrho(1-s)$

The generalisation of the above yields the following.

Theorem II. *If $f(x)$ is a function of the form*

$$f(x) = \frac{1}{2\pi i}\int_{c+i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\lambda-\tfrac{1}{2}s)\psi(s)x^{-s}ds,$$

where $0<c<1$ and $\psi(s) = \psi(1-s)$, then the function

$$g(x) = \int_0^{1/x} Q\left(\log\frac{1}{xy}\right)f(y)dy$$

is R_ν, provided that $Q(x)$ is given by (4.1).

The proof of this is exactly similar to that of Theorem I proved above.

Corollary 1. *When $\lambda = \nu$ the theorem is reducible to the one due to B. Mohan given above.*

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BENDING OF CERTAIN CLAMPED THIN CURVILINEAR ELASTIC PLATES NORMALLY LOADED OVER A CONCENTRIC CIRCLE

By

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In this paper complex variable methods are used to derive exact solutions in closed forms for the small deflexions of certain thin elastic plates subject to normal loading over a concentric circle, the loading considered being symmetrical with respect to the centre of the circle. The isotropic plates considered are bounded by certain types of curvilinear edges along which the plates are rigidly clamped. The circular plate and a plate bounded by an inverse of an ellipse with respect to its centre are included as special cases.

1. **Introduction.** Problems concerning the transverse flexure of thin clamped or simply supported elastic plates having various shapes and subject to various distributions of normal loading have been considered by several investigators. In a series of papers Bassali (1956a, 1956b) and Bassali and Dawoud (1956) discussed the bending of thin circular plates subject to different types of normal loading distributed over the area of an eccentric circle; methods of complex function theory being used to obtain the solution. Sen (1942) used curvilinear co-ordinates to solve the problems of bending of clamped and uniformly loaded thin elastic plates bounded by quartic curves having the forms of the cardioid, the loop of a lemniscate of Bernoulli, the inverse of an ellipse with respect to its centre and inverse of an ellipse with respect to its focus (the elliptic limacon). The solution appropriate to a concentrated load at any point of a clamped elliptic plate has been obtained in series form by Sengupta (1948, 1949). A simple method of determining the deflexion of certain types of elastic plates, subject to transverse isolated loads at certain points was given by Sen (1934) The same method was applied by Das (1950) to establish closed expressions for the deflexion at any point of (a) a clamped plate bounded by the inverse of an ellipse with respect to its centre and loaded by a concentrated force at its centre, (b) a clamped plate bounded by an elliptic limacon and subject to an isolated load at the focus. The solution for a clamped regular polygonal plate with n sides and subject to uniform loading distributed over the entire plate has been given in a closed form by Stevenson (1943). When the same plate is under an isolated load at any point or loaded over a concentric circular area the solutions have been obtained by Dawoud (1950) or Bassali (1958) respectively. The solutions for certain clamped thin elastic slabs with curvilinear boundaries, subject to transverse concentrated forces or couples applied at arbitrary or specified points have been obtained

by Bassali (1959). Mukhelishvili's method of dealing with biharmonic equation has been applied by Gray (1952) to the analysis of clamped thin elastic slabs under concentrated loads.

In this paper complex variable methods are used to find closed expressions for the complex potentials and deflexion at any point of a curvilinear plate, subject to normal loading overa concentric circular area. The load taken is symmetrical with respect to the centre of the circle. The plate, taken in the z-plane can be mapped on the region inside the unit circle in the ζ-plane by the mapping function

$$z = c\zeta/(1+m\zeta^{p+1}), \quad (c > 0, \, |m| \leq 1/p),$$

where c, m and p are real constants, p being a positive integer. The tentative method of solution adopted here is to assume suitable forms for the complex potentials and we then show that they fit the conditions given, provided certain constants involved take on certain definite values. This method has been extensively used by Stevenson (1942, 1943) and Bassali (1958, 1959).

2. **Formulation.** Consider a clamped thin isotropic plate bounded by a closed curve C_p. When the middle plane of a normally loaded plate is assumed to be in the complex plane $z = x+iy$ its deflection w normal to the plane of the plate is governed by the biharmonic equation

$$\nabla^4 w = 16 \frac{\partial^4 w}{\partial z^2 \partial \bar{z}^2} = -p(z, \bar{z})/D, \tag{2.1}$$

where $p(z, \bar{z})$ is the transverse load intensity and $D = 2Eh^3/3(1-\nu^2)$ is the flexural rigidity in which E and ν are Young's modulus and Poisson's ratio respectively, for the material of the plate, $2h$ being the thickness of the plate and bars used to denote conjugate complex quantities.

The general solution of (2.1) may be expressed in terms of two analytic functions $\varphi(z)$, $\psi(z)$ and a particular integral $W(z, \bar{z})$ corresponding to the load intensity $p(z, \bar{z})$:

$$w = \bar{z}\varphi(z) + z\bar{\varphi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z}) + W(z, \bar{z}), \tag{2.2}$$

$$\chi(z) = \int \psi(z) dz.$$

$W(z, \bar{z})$ may be taken as zero if z lies in an unloaded region.

Boundary conditions that the boundary C_p is clamped may be taken as

$$w = 0, \ \frac{\partial w}{\partial n} = 0 \text{ along } C_p, \tag{2.3}$$

where $\partial/\partial n$ denotes differentiation along the outward drawn normal. It is easily seen that these conditions are equivalent to

$$w = 0, \ \frac{\partial w}{\partial z} = 0 \text{ along } C_p. \tag{2.4}$$

Using (2.2), conditions (2.4) can be written as

$$\bar{z}\varphi(z)+z\bar{\varphi}(\bar{z})+\chi(z)+\bar{\chi}(\bar{z})+W(z,\bar{z}) = 0 \text{ along } C_p \tag{2.5}$$

$$\bar{z}\varphi'(z)+\bar{\varphi}(\bar{z})+\psi(z)+\frac{\partial W}{\partial z} = 0 \text{ along } C_p \tag{2.6}$$

where accents denote differentiation with respect to the argument.

3. Clamped thin curvilinear plate normally loaded over a concentric circle. The transformation

$$z = \omega(\zeta) = \frac{c\zeta}{1+m\zeta^{p+1}}, \quad \zeta = \varrho e^{i\theta}, \tag{3.1}$$

$$c>0, \quad |m| \leq 1/p,$$

where c, m are real constants and p a positive integer, maps the interior of a closed curve C_p in the z-plane on the interior of a unit circle γ in a complex ζ-plane. The parametric equations of the closed curve C_p are given by

$$\frac{x}{c} = \frac{\cos\theta+m\cos p\theta}{1+m^2+2m\cos(p+1)\theta}, \quad \frac{y}{c} = \frac{\sin\theta-m\sin p\theta}{1+m^2+2m\cos(p+1)\theta}. \tag{3.2}$$

It is easily seen that C_p is the inverse with respect to the circle $|z| = c$ of a closed curve Γ_p defined by

$$x = c(\cos\theta+m\cos p\theta), \quad y = c(\sin\theta-m\sin p\theta). \tag{3.3}$$

The curve C_1 corresponding to $p = 1$ is the inverse of the ellipse $x^2/a^2+y^2/b^2 = 1$ with respect to a concentric circle of radius d, where in this case

$$z = \frac{c\zeta}{1+m\zeta^2}, \quad (0 \leq m \leq 1),$$

$$c = \frac{2d^2}{a+b}, \quad m = \frac{a-b}{a+b}. \tag{3.4}$$

The resulting figure in the z-plane is called the lemniscate of Booth and is shown in Fig. 1 on p. 70. When m is almost equal to unity it differs little from that produced by two touching circles of equal radii [Muskhelishvili (1949), p. 180; Sokolnikoff (1946), p. 176, 183]. The limiting case in which $b \to 0$ and $m \to 1$ is worth consideration. In this case the plate in the z-plane is infinite and is clamped along the two semi infinite lines $x \geq \frac{1}{2}c$, $y = 0$ and $x \leq -\frac{1}{2}c$, $y = 0$ represented respectively by AB (or AD) and $A'B$ (or $A'D$) in Fig. 1, these lines being the only boundaries of the plate.

When $m = 1/p (p>1)$ the curve C_p is the inverse with respect to a concentric circle of a star shaped hypocycloid Γ_p with $p+1$ cusps [Muskhelishvili (1949), p. 178]. When $m = -2/p(p+1)$ the closed curve Γ_p approximates to a regular rectilinear polygon

with $p+1$ sides and the corresponding curve C_p is therefore approximately bounded by $p+1$ circular arcs each of which passes through the centre and two consecutive vertices of a regular rectilinear polygon with $p+1$ sides. The curves C_p and Γ_p corresponding to $p=2$, $m=-\frac{1}{3}$, $p=3$, $m=-\frac{1}{6}$ and $p=7$, $m=-\frac{1}{28}$ are shown in Figs. 2, 3 and 4 respectively.

Let the plate be normally loaded over the area of a concentric circle C of radius $a\leq\frac{c}{1+|m|}$ and clamped along the boundary. Let 1 and 2 refer to the two regions inside C and between C and C_p respectively. Let the transverse load intensity p on the plate be given by

$$p_1 = p_0\, r^{n-2} = p_0(z\bar{z})^{\frac{n-2}{2}}, \quad n \geq 2 \text{ over } 1$$

$$p_2 = 0 \text{ over } 2 \tag{3.5}$$

For $n=2$ we have uniform loading over 1.

The particular integrals W_1 and W_2 may be taken as

$$W_1(z,\bar{z}) = -\,p_0(z\bar{z})^{n'/2}/n^2n'^2D, \quad (n'=n+2),$$

$$W_2(z,\bar{z}) = 0. \tag{3.6}$$

Let the complex potentials in the regions 1 and 2 be denoted by $\varphi_1(z)\equiv\varphi_1(\zeta)$, $\chi_1(z)\equiv\chi_1(\zeta)$ and $\varphi_2(z)\equiv\varphi_2(\zeta)$, $\chi_2(z)\equiv\chi_2(\zeta)$ respectively. Since $W_2(z,\bar{z})=0$, the boundary conditions (2.5) and (2.6) are transformed to

$$\bar{\omega}(\bar{\sigma})\varphi_2'(\sigma)+\omega(\sigma)\bar{\varphi}_2(\bar{\sigma})\chi+\chi_2'(\sigma)+\bar{\chi}_2(\bar{\sigma})=0 \tag{3.7}$$

$$\bar{\omega}(\bar{\sigma})\varphi_2'(\sigma)+\omega'(\sigma)\bar{\varphi}_2(\bar{\sigma})+\chi_2'(\sigma)=0 \tag{3.8}$$

where $\sigma=e^{i\theta}$ is the value of ζ on γ.

It has been shown by Dawoud (1954) that when the two regions are continuously and differently loaded, the necessary and sufficient conditions for the physical continuity across C can be taken as

$$w_1=w_2,\ \frac{\partial w_1}{\partial z}=\frac{\partial w_2}{\partial z},\ \frac{\partial^2 w_1}{\partial z^2}=\frac{\partial^2 w_2}{\partial z^2},$$

$$\frac{\partial^3 w_1}{\partial z^2\partial\bar{z}}=\frac{\partial^3 w_2}{\partial z^2\partial\bar{z}} \text{ along } C \tag{3.9}$$

Now using (2 2) and (3 6) we find that the continuity condition (2 9) are satisfied if

$$\left[\bar{z}\varphi(z)+z\bar{\varphi}(\bar{z})+\chi(z)+\bar{\chi}(\bar{z})\right]_2^1=\frac{8\alpha a^2}{nn'^2} \tag{3.10}$$

$$\left[\bar{z}\varphi'(z)+\bar{\varphi}(\bar{z})+\chi'(z)\right]_2^1=\frac{4\alpha a^2}{nn'z} \tag{3.11}$$

$$\left[\varphi'(z)+\bar{\varphi}'(\bar{z})\right]_2^1=\frac{2\alpha}{n} \tag{3.12}$$

$$\left[\varphi''(z)\right]_2^1=\frac{\alpha}{z} \tag{3.13}$$

where

$$\alpha=P_0/16\pi D, \tag{3.14}$$

$$P_0=2\pi p_0a^n/n. \tag{3.15}$$

It may be noted that P_0 is the total load on the plate.

As tentative solutions for these continuity conditions we assume that

$$\left[\varphi(z)\right]_2^1=z\left(A+B\log\frac{z}{a}\right) \tag{3.16}$$

$$\left[\chi(z)\right]_2^1=E+F\log\frac{z}{a} \tag{3.17}$$

where A, B, E and F are real constants. Substituting from (3.16) and (3.17) in (3.12) and (3 13) and equating the coefficients of the same powers of z on both sides we get

$$A=\alpha\left(\frac{1}{n}-1\right),\quad B=\alpha \tag{3.18}$$

Hence we have
$$\left[\varphi(z)\right]_2^1=\alpha z\left(\frac{1}{n}-1+\log\frac{z}{a}\right) \tag{3.19}$$

Substituting from (3.19) and (3 17) in (3 10) and (3 11) and equating the coefficients of the same powers of z on both sides we get

$$E=\alpha a'^2\left(\frac{1}{n'}+1\right),\quad F=\alpha a'^2, \tag{3.20}$$

where

$$a'^2=na^2/n'. \tag{3.21}$$

Hence we have
$$\left[\chi'(z)\right]_2^1=\alpha a'^2\left(\frac{1}{n'}+1+\log\frac{z}{a}\right) \tag{3.22}$$

Let us assume tentatively that

$$\varphi_2(z) \equiv \varphi_2(\zeta) = \alpha z(a_0 + a_p\zeta^{p+1} - \log \zeta), \tag{3.23}$$

$$\chi_2(z) \equiv \chi_2(\zeta) = \alpha\left[\frac{cz}{\zeta}(b_0 + b_p\zeta^{p+1}) - a'^2 \log \zeta\right], \tag{3.24}$$

where a_0, b_0, a_p and b_p are real constants.

Now from (3.19), (3.22), (3.23) and (3.24) we easily find that

$$\varphi_1(z) \equiv \varphi_1(\zeta) = \alpha z\left(a'_0 + a'_p\zeta^{p+1} + \log \frac{z}{a\zeta}\right), \tag{3.25}$$

$$\chi_1(z) \equiv \chi_1(\zeta) = \alpha\left[\frac{cz}{\zeta}(b'_0 + b'_p\zeta^{p+1}) + a'^2 \log \frac{z}{a\zeta}\right], \tag{3.26}$$

where

$$\left.\begin{aligned} a_0' &= a_0 + (1/n) - 1, \\ b_0' &= b_0 + a'^2\{(1/n') + 1\}/c^2, \\ a'_p &= a_p + m\{(1/n) - 1\}, \\ b'_p &= b_p. \end{aligned}\right\} \tag{3.27}$$

It is clear from (3.23) – (3.26) and (3.1) that φ_1, χ_1 and φ_2, χ_2 are regular in their respective regions.

Now we have to determine the real constants a_0, b_0 a_p and b_p from the boundary conditions (3.7) and (3.8). Substituting from (3.1), (3.23) and (3.24) in (3.7) we find

$$2a_0 + a_p(\sigma^{p+1} + \sigma^{-p-1}) + (b_0 + b_p\sigma^{p+1})(1 + m\sigma^{-p-1}) + (b_0 + b_p\sigma^{-p-1})(1 + m\sigma^{p+1}) = 0.$$

Equating the coefficients of various powers of σ to zero we obtain

$$\left.\begin{aligned} a_0 + b_0 + mb_p &= 0 \\ a_p + b_p + mb_0 &= 0 \end{aligned}\right\} \tag{3.28}$$

Similarly substituting from (3.1), (3.25) and (3.26) in (3.8) and simplifying we get

$$\begin{aligned} [a_0 + (p+2)a_p\sigma^{p+1}](1 + m\sigma^{p+1})\sigma^{-1} - m(p+1)(a_0 + a_p\sigma^{p+1})\sigma^p + (a_0 + a_p\sigma^{-p-1})(1 - mp\sigma^{p+1})\sigma^{-1} \\ + (p+1)(b_p - mb_0)(1 + m\sigma^{-p-1})\sigma^p \end{aligned}$$

$$= \left[1 + (1+2m^2)\frac{a'^2}{c^2}\right]\sigma^{-1} + m\frac{a'^2}{c^2}\sigma^{-p-2} + \left[1 + (2 + m^2)\frac{a'^2}{c^2}\right]m\sigma^p + m^2\frac{a'^2}{c^2}\sigma^{2p+1}$$

which is satisfied if

$$\left.\begin{aligned} 2a_0 + m(p+1)(b_p - mb_0) - mpa_p &= 1 + (1 + 2m^2)\frac{a'^2}{c^2} \\ 2mpa_0 - (p+1)(b_p - mb_0) - (p+2)a_p &= -m\left[1 + (2 + m^2)\frac{a'^2}{c^2}\right] \\ a_p &= m\frac{a'^2}{c^2} \end{aligned}\right\} \tag{3.29}$$

The five equations given by (3.28) and (3.29) are satisfied for the following values of a_0, b_0, a_p and b_p:

$$\left.\begin{aligned} a_0 &= [1-m^2+k\{1+2(p+1)m^2-m^4\}]/\Delta, \\ b_0 &= -[1+k\{1+(2p+1)m^2\}]/\Delta, \\ a_p &= km, \\ b_p &= m[1-k(1-m^2)]/\Delta, \end{aligned}\right\} \tag{3.30}$$

where

$$k = \frac{a'^2}{c^2}, \quad \Delta = 2(1+pm^2). \tag{3.31}$$

Now from (3.27) and (3.30) we have

$$\left.\begin{aligned} a'_0 &= \frac{1}{n} - [1+(2b+1)m^2+k\{1+2(p+1)m^2-m^4\}]/\Delta, \\ b'_0 &= \frac{k}{n'} - [1-k(1-m^2)]/\Delta, \\ a'_p &= m\left(\frac{1}{n}-1+k\right), \\ b'_p &= m[1-k(1-m^2)]/\Delta. \end{aligned}\right\} \tag{3.32}$$

The boundary conditions along C_p and the continuity conditions across C are thus satisfied and the complex potentials are now completely determined by (3.23) – (3.26) and (3.30) – (3.32).

The deflection at any point of the regions 1 and 2 is given by the following expressions:

$$w_1 = 2\alpha\left[r^2\left\{a'_0+\frac{b'_0}{\varrho^2}+mb'_p\varrho^{2p}+\left(\frac{mk}{n'\varrho^2}+a'_p\right)\varrho^p\cos(p+1)\theta\right\}+(r^2+a'^2)\log\frac{r}{a\varrho}-\frac{4r^{n'}}{nn'^2a^n}\right], \tag{3.33}$$

$$w_2 = 2\alpha\left[r^2\left\{a_0+\frac{b_0}{\varrho^2}+mb_p\varrho^{2p}+mk\left(1-\frac{1}{\varrho^2}\right)\varrho^p\cos(p+1)\theta\right\}-(r^2+a'^2)\log\varrho\right], \tag{3.34}$$

where

$$r^2 = z\bar{z}.$$

The deflection w_0 at the centre of the plate is given by

$$w_0 = 2X_1(0) = 2\alpha\left(C^2b'_0+a'^2\log\frac{c}{a}\right)$$

Substituting the values of α and b'_0 from (3.14) and (3.32) we have

$$w_0 = \frac{P_0 c^2}{16\pi D}\left[2k\left(\frac{1}{n'} + \log\frac{c}{a}\right) - \frac{1-k(1-m^2)}{1+pm^2}\right]. \tag{3.35}$$

The bending moments, twisting moments M_ρ, M_θ, $M_{\rho\theta}$ and shearing forces Q_ρ, Q_θ all per unit length, at any point of the plate may be computed by using the standard formulae. We know that for unloaded region of the plate

$$M_\rho + M_\theta = -8D(1+\nu)[\varphi'_2(z) + \bar{\varphi}'_2(\bar{z})], \tag{3.36}$$

and since along the clamped boundary $\nu M_\rho = M_\theta$, we have

$$[M_\rho]_{\zeta=\sigma} = -8D\left[\frac{\varphi'_2(\zeta)}{\omega'(\zeta)} + \frac{\bar{\varphi}'_2(\bar{\zeta})}{\bar{\omega}'(\bar{\zeta})}\right]_{\zeta=\sigma} \tag{3.37}$$

Now substituting the values of $\varphi_2(\zeta)$ and $\omega(\zeta)$ from (3.23) and (3.1) and using (3.38) and (3.14) we get

$$[M_\rho]_{\zeta=\sigma} = \frac{P_0}{2\pi[1+m^2p'+2mp'\cos(p+1)\theta]}\,[1+m^2(1+p\,k+pa_0)$$

$$+m\{p+3-2a_0-p'k(1+m^2)\}\cos(p+1)\theta + m^2k\cos(2p+2)\theta] \tag{3.38}$$

where $$p' = p+2.$$

For $m = 0$, the previous results reduced to those already found by Bassali and Dawoud (1956) for a clamped circular plate normally and symmetrically loaded over a concentric circle.

The solution corresponding to a concentrated transverse load P_0 acting at the centre of the plate, can be derived from the foregoing results by making the radius a of the loaded area approach zero. Now taking $a = a' = k = 0$, we get the following expressions for the complex potentials and deflexion for this case:

$$\varphi(z) = \tfrac{1}{2}\alpha z\left[\frac{1-m^2}{1+m^2p} - 2\log\zeta\right],$$

$$\psi(z) = -\frac{\alpha c^2}{2(1+m^2p)}\,\frac{1-m\zeta^{p+1}}{1+m\zeta^{p+1}},$$

$$w = \alpha r^2\left[\frac{1-\varrho^2-m^2(1-\varrho^{2p})}{1+m^2p} - 2\log\varrho\right], \tag{3.39}$$

which agrees with the solution derived by Bassali (1959).

In the limiting case in which $p = 1$ and m tends to unity the plate becomes infinite

and is bounded by and clamped along the semi-infinite lines AB and $A'D$ of Fig. 1. In this case taking $m = p = 1$ we get the following expressions for w_1 and w_2:

$$w_1 = \frac{P_0}{8\pi D}\left[r^2\left\{\frac{1}{n} - 1 + k + \frac{4k-n'}{4n'\varrho^2} + \frac{1}{4}\varrho^2\right.\right.$$

$$\left.\left.+\left(\frac{k}{n'\varrho^4} \quad \frac{1-n+nk}{n}\right)\varrho\cos 2\theta\right\} - (r^2+a'^2)\log\varrho\right], \tag{3.40}$$

$$w_2 = \frac{P_0}{8\pi D}\left[r^2\left\{k - \frac{1+4k}{4\varrho^2} + \tfrac{1}{4}\varrho^2\right.\right.$$

$$\left.\left.+ k\left(1-\frac{1}{\varrho^2}\right)\varrho\cos 2\theta\right\} - (r^2+a'^2)\log\varrho\right]. \tag{3.41}$$

In case the same plate is acted upon by a concentrated force at the centre of the plate, the expression for the deflection may be derived from (3.41) by taking $a' = k = 0$. In this case we get

$$w = \frac{P_0}{8\pi D}\, r^2\left[\tfrac{1}{4}\left(\varrho^2 - \frac{1}{\varrho^2}\right) - \log\varrho\right], \tag{3.42}$$

which is a particular case of relation (12) of Dean (1953).

When the plate is of the shape of a lemniscate of Booth and is acted upon by a concentrated force P_0 at the centre, the expression for deflection can be obtained by taking $p = 1$ in the last relation (3.39) and hence putting $p = 1$ and substituting the value of α from (3.14) in (3.39) we get,

$$w = \frac{P_0}{16\pi D}\,\frac{r^2}{1+m^2}\left[1 - m^2 - \frac{1}{\varrho^2} + m^2\varrho^2 - (1+m^2)\log\varrho\right]. \tag{3.43}$$

Now putting $\qquad m^{\frac{1}{2}}\zeta = e^{i(u+iv)}, \quad (0 \le m \le 1),$

in (3.1) we see that the transformation (3.1) takes the form

$$z = q\sec(u+iv)$$

where $\qquad q = c/2m^{\frac{1}{2}}, \quad u = \theta, \quad m^{\frac{1}{2}}\varrho = e^{-v}. \tag{3.44}$

Now setting $\qquad m = e^{-2\beta}\ (0 \le \beta \le \infty)$ we have

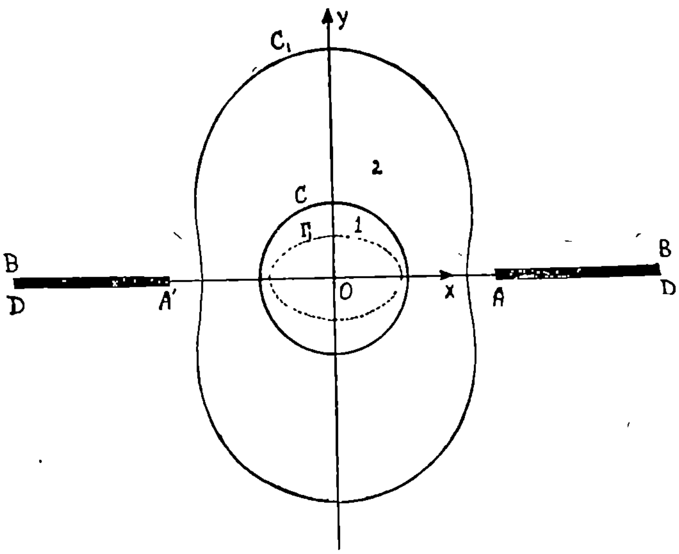
$$\varrho = e^{\beta - v}, \quad c = 2qe^{-\beta}.$$

It is now evident that straight line $v = \beta$ in (u,v) plane corresponds to the boundary γ of the unit circle in the ζ-plane and hence to the boundary C_1 of the lemniscate of Booth in the z-plane.

With the above substitutions (3.43) reduces to

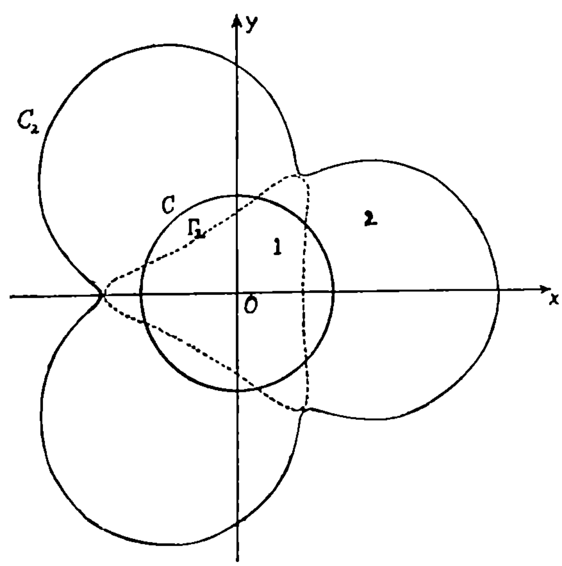
$$w = -\frac{P_0 q^2}{8\pi D}\left[\frac{2(\beta - v) + (\sinh 2v - \sinh 2\beta)/\cosh 2\beta}{\cosh 2v + \cos 2u}\right] \tag{3.45}$$

which coincides with the solution derived by Das (1950) using the method of Sen (1934), on noting the difference in notation.



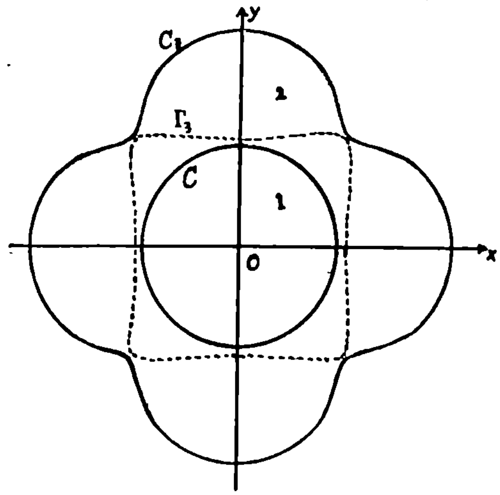
$p=1$, $m=\frac{1}{4}$

Fig. 1



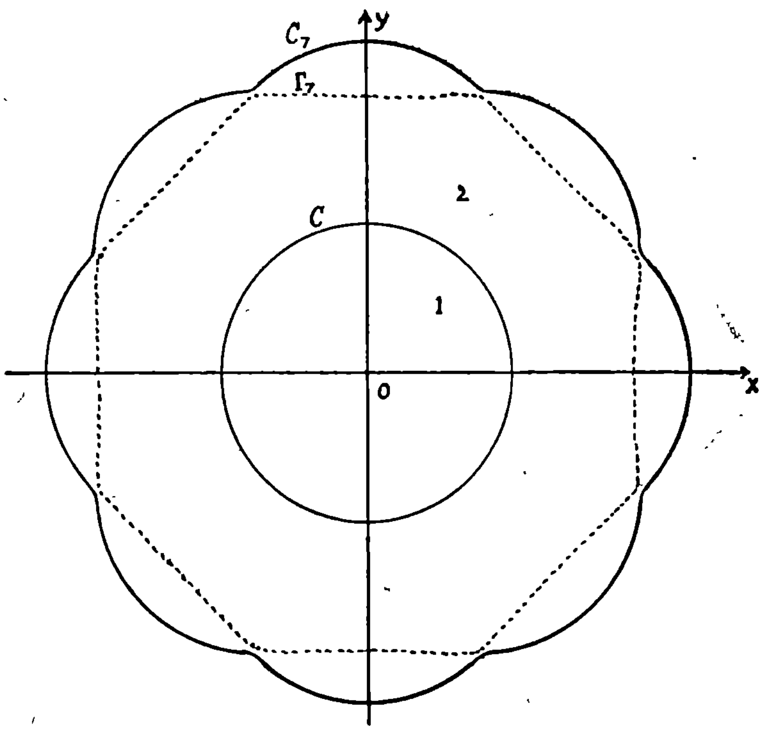
$p=2$, $m=-\frac{1}{3}$

Fig. 2



$p = 3, \; m = -\frac{1}{6}$

Fig. 3



$p = 7, \; m = -\frac{1}{28}$

Fig. 4

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ON THE MEANS OF AN ENTIRE FUNCTION AND ITS DERIVATIVES

By

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1. Let $f(z)$ be an entire function of order ϱ and lower order λ and let

$$\mu_\delta(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \tag{1.1}$$

$$\mu_\delta(r,f^{(n)}) = \frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^\delta d\theta \tag{1.2}$$

$$m_{\delta,k}(r,f) = \frac{1}{\pi r^{k+1}}\int_0^r\int_0^{2\pi} |f(xe^{i\theta})|^\delta x^k dx d\theta \tag{1.3}$$

$$m_{\delta,k}(r,f^{(n)}) = \frac{1}{\pi r^{k+1}}\int_0^r\int_0^{2\pi} |f^{(n)}(xe^{i\theta})|^\delta x^k dx d\theta, \tag{1.4}$$

where $f^{(n)}(z)$ is the n^{th} derivative of $f(z)$. It is known that [Rahman (1956), p. 193]

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu_\delta(r,f)}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix};$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_{\delta,k}(r,f)}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

It is also known that [Srivastava (1959), p 230]

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\left[r\left\{\frac{\mu_\delta(r,f^{(1)})}{\mu_\delta(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

In this paper we would extend this result to $0 < \delta < 1$ and to $m_{\delta,k}(r)$. We have also obtained a few properties of $m_{\delta,k}(r)$. The results are given in the form of theorems.

2 **Theorem 1**. *For every entire function $f(z)$, other than a polynomial, of order ϱ and lower order λ,*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\left[r\left\{\frac{m_{\delta,k}(r,f^{(1)})}{m_{\delta,k-\delta}(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix},$$

where $k > \delta > 1$

The proof of this theorem is based on the following two lemmas :

Lemma 1. *For every entire function $f(z)$, other than a polynomial,*

$$m_{\delta,k}(r,f^{(1)}) \geq m_{\delta,k-\delta}(r,f)\left[\frac{\log m_{\delta,k-\delta}(r,f)}{\delta r \log r}\right]^{\delta},$$

for all $\qquad r > r_0 = r_0(f)$ *and* $k > \delta > 1$

Proof. We have

$$m_{\delta,k}(r,f^{(1)}) = \frac{1}{\pi r^{k+1}} \int_0^r \int_0^{2\pi} |f^{(1)}(xe^{i\theta})|^{\delta} x^k \, dx \, d\theta$$

$$= \frac{1}{\pi r^{k+1}} \int_0^r \int_0^{2\pi} \lim_{\epsilon \to 0+} \left| \frac{f(xe^{i\theta}) - f(\overline{x - x\epsilon}\, e^{i\theta})}{\epsilon x e^{i\theta}} \right|^{\delta} x^k \, dx \, d\theta$$

$$\geq \frac{1}{\pi r^{k+1}} \lim_{\epsilon \to 0+} \int_0^r \int_0^{2\pi} \left\{ \frac{|f(xe^{i\theta})| - |f(\overline{x - x\epsilon}\, e^{i\theta})|}{\epsilon x} \right\}^{\delta} x^k \, dx \, d\theta \tag{2.2}$$

By Minkowski's inequality [Titchmarsh, 1939, p. 384]

$$\left[\int_0^{2\pi} (|f(xe^{i\theta})| - |f(\overline{x - x\epsilon}\, e^{i\theta})|)^{\delta} d\theta\right]^{\frac{1}{\delta}} \geq \left[\left(\int_0^{2\pi} |f(xe^{i\theta})|^{\delta} d\theta\right)^{\frac{1}{\delta}} - \left(\int_0^{2\pi} |f(\overline{x - x\epsilon}\, e^{i\theta})|^{\delta} d\theta\right)^{\frac{1}{\delta}}\right].$$

Hence,

$$m_{\delta,k}(r,f^{(1)}) \geq \lim_{\epsilon \to 0+} \frac{1}{\pi r^{k+1}} \left[\int_0^r \left\{\left(\int_0^{2\pi} |f(xe^{i\theta})|^{\delta} d\theta\right)^{\frac{1}{\delta}} - \left(\int_0^{2\pi} |f(\overline{x - x\epsilon}\, e^{i\theta})|^{\delta} d\theta\right)^{\frac{1}{\delta}}\right\}^{\delta} \frac{x^{k-\delta}}{\epsilon^{\delta}} \, dx\right]$$

$$= \lim_{\epsilon \to 0+} \frac{1}{\pi r^{k+1} \epsilon^{\delta}} \left[\int_0^r \left\{\left(x^{k-\delta} \int_0^{2\pi} |f(xe^{i\theta})|^{\delta} d\theta\right)^{\frac{1}{\delta}} - \left(x^{k-\delta} \int_0^{2\pi} |f(\overline{x - x\epsilon}\, e^{i\theta})|^{\delta} d\theta\right)^{\frac{1}{\delta}}\right\}^{\delta} dx\right]$$

Again, using Minkowski's inequality, we obtain

$$m_{\delta,k}(r,f^{(1)}) \geq \lim_{\epsilon \to 0+} \frac{1}{\pi r^{k+1} \epsilon^{\delta}} \left[\left\{\int_0^r \int_0^{2\pi} |f(xe^{i\theta})|^{\delta} x^{k-\delta} dx \, d\theta\right\}^{\frac{1}{\delta}} - \left\{\int_0^r \int_0^{2\pi} |f(\overline{x - x\epsilon}\, e^{i\theta})|^{\delta} x^{k-\delta} dx \, d\theta\right\}^{\frac{1}{\delta}}\right]^{\delta}$$

$$\geq \lim_{\epsilon \to 0+} \frac{1}{\pi r^{k+1} \epsilon^{\delta}} \left[\left\{\pi r^{k-\delta+1} m_{\delta,k-\delta}(r,f)\right\}^{\frac{1}{\delta}} - \left\{\pi r^{k-\delta+1} (1-\epsilon)^{k-\delta+1} m_{\delta,k-\delta}(\overline{r - r\epsilon}, f)\right\}^{\frac{1}{\delta}}\right]^{\delta}$$

$$\geq \lim_{\epsilon \to 0+} \left[\frac{\{m_{\delta,k-\delta}(r,f)\}^{\frac{1}{\delta}} - \{m_{\delta,k-\delta}(\overline{r - r\epsilon}, f)\}^{\frac{1}{\delta}}}{\epsilon r}\right]^{\delta}$$

Now, take $g(r) = \dfrac{\log m_{\delta,k-\delta}(r,f)}{\log r}$, then since $g(r)$ is a positive indefinitely increasing function of r for $r > r_0 = r_0(f)$, in fact $\log m_{\delta,k-\delta}(r,f)$ is a convex function of $\log r$, and so we have

$$m_{\delta,k}(r,f^{(1)}) \geq \left[\frac{1}{\delta}\left\{m_{\delta,k-\delta}(r,f)\right\}^{\frac{1}{\delta}-1} r^{q(r)-1} g(r)\right]^{\delta}$$

$$= \left[\frac{1}{\delta}\left\{m_{\delta,k-\delta}(r,f)\right\}^{\frac{1}{\delta}-1}\left\{\frac{\log m_{\delta,k-\delta}(r,f)}{r \log r}\right\}\left\{m_{\delta,k-\delta}(r,f)\right\}\right]^{\delta},$$

$$i.e.\ m_{\delta,k}(r,f^{(1)}) \geq m_{\delta,k-\delta}(r,f)\left[\frac{\log m_{\delta,k-\delta}(r,f)}{\delta r \log r}\right]^{\delta}.$$

$\log m_{\delta,k}(r,f)$ is a convex function of $\log r$ can be proved in the following manner:

$$\frac{d(\log m_{\delta,k}(r))}{d(\log r)} = r\left[\frac{\dfrac{1}{\pi r^{k+1}}\, r^{k}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^{\delta} d\theta - \dfrac{k+1}{\pi r^{k+2}}\displaystyle\int_0^{r}\int_0^{2\pi} |f(xe^{i\theta})|^{\delta} x^{k} dx\, d\theta}{m_{\delta,k}(r)}\right]$$

$$= \left[\frac{\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^{\delta} d\theta - \dfrac{k+1}{\pi r^{k+1}}\displaystyle\int_0^{r}\int_0^{2\pi} |f(xe^{i\theta})|^{\delta} x^{k} dx\, d\theta}{m_{\delta,k}(r)}\right]$$

$$= \left[\frac{2\mu_{\delta}(r)}{m_{\delta,k}(r)} - (k+1)\right].$$

which increases with r for $r > r_0$, since $\mu_\delta(r)$ is a convex function of $m_{\delta,k}(r)$. [Rahman (1956), pp. 194, 195].

Hence $$\frac{d^2(\log m_{\delta,k}(r))}{d(\log r)^2} > 0 \qquad \text{for } r > r_0.$$

Lemma 2 *If $f(z)$, other than a polynomial, is regular for $|z| \leq r$ and if $z = xe^{i\theta}$, $0 < x < r$, and $k > \delta > 1$, then*

$$m_{\delta,k}(r,f^{(1)}) \leq \frac{(1+\varepsilon)^{\delta}}{(2\pi)^{\delta-1}}\left[\frac{\log \mu_{\delta}(r,f)}{\log r}\right]^{\delta} \frac{2}{r^{\delta}\, r^{k-\delta+1}} \int_0^{r} \mu_{\delta}(x + x\varepsilon) x^{k-\delta} dx.$$

Proof. By Cauchy's theorem

$$|f^{(1)}(z)| \leq \frac{1}{2\pi}\, \frac{r}{(r-x)^2}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta, \quad 0 < x < r, \quad |z| = x.$$

Also, by Holder's inequality, [Titchmarsh, 1939, p. 382]

$$|f^{(1)}(z)|^{\delta} \leq \left(\frac{1}{2\pi}\, \frac{r}{(r-x)^2}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta\right)^{\delta}$$

$$\leq \frac{r^{\delta}}{(2\pi)^{\delta-1}(r-x)^{2\delta}}\, \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^{\delta} d\theta$$

$$= \frac{r^{\delta}}{(2\pi)^{\delta-1}(r-x)^{2\delta}}\mu_{\delta}(r,f).$$

Now, taking $r = x + x\left(\frac{\log x}{\log \mu_\delta(x,f)}\right)^{\frac{1}{k}}$ and integrating the above inequality with respect to θ between the limits 0 and 2π, we get

$$\mu_\delta(x,f^{(1)}) \leq \frac{1}{(2\pi)^{\delta-1}}\mu_\delta\left[x + x\left\{\frac{\log x}{\log \mu_\delta(x,f)}\right\}^{\frac{1}{k}}\right]\left[\frac{\log \mu_\delta(x,f)}{x \log x}\right]^{\delta}\left[1+\left\{\frac{\log x}{\log \mu_\delta(x,f)}\right\}^{\frac{1}{k}}\right]^{\delta},$$

But $\left(\frac{\log x}{\log \mu_\delta(x,f)}\right)^{\frac{1}{k}} \to 0$ as $x \to \infty$, since $\log \mu_\delta(x,f)$ is a steadily increasing convex function of $\log x$. Hence for any $\epsilon > 0$, we can find an $x_0 = x_0(\epsilon,f)$, such that for $x > x_0$

$$\left(\frac{\log x}{\log \mu_\delta(x,f)}\right)^{\frac{1}{k}} < \epsilon.$$

Hence, $$\mu_\delta(x,f^{(1)}) \leq \frac{1}{(2\pi)^{\delta-1}}\mu_\delta(x+x\epsilon)\left[\frac{\log \mu_\delta(x,f)}{x \log x}\right]^{\delta}[1+\epsilon]^{\delta}.$$

Integrating the above inequality after multiplying by x^k with respect to x between the limits o and r, we get

$$\int_0^r x^k \mu_\delta(x,f^{(1)})dx \leq \frac{(1+\epsilon)^\delta}{(2\pi)^{\delta-1}}\int_0^r \left\{\frac{\log \mu_\delta(x,f)}{x \log x}\right\}^{\delta}\mu_\delta(x+x\epsilon)x^k dx$$

$$\leq \frac{(1+\epsilon)^\delta}{(2\pi)^{\delta-1}}\left\{\frac{\log \mu_\delta(r,f)}{\log r}\right\}^{\delta}\int_0^r \mu_\delta(x+x\epsilon)x^{k-\delta}dx.$$

Hence, $$m_{\delta,k}(r,f^{(1)}) \leq \frac{(1+\epsilon)^\delta}{(2\pi)^{\delta-1}}\left\{\frac{\log \mu_\delta(r,f)}{\log r}\right\}^{\delta}\frac{2}{r^{\delta}r^{k-\delta+1}}\int_0^r \mu_\delta(x+x\epsilon)x^{k-\delta}dx.$$

Proof of Theorem 1. We have from Lemma 1.

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\left[r\left\{\frac{m_{\delta,k}(r,f^{(1)})}{m_{\delta,k-\delta}(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} \geq \frac{\varrho}{\lambda},$$

also from lemma 2.

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\left[r\left\{\frac{m_{\delta,k}(r,f^{(1)})}{m_{\delta,k-\delta}(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} \leq \frac{\varrho}{\lambda};$$

and hence the result.

Corollary. If we take out the least value of $1/x^\delta$ from the integral in (2.2) and proceed in the same manner as in lemma 1 we obtain

$$m_{\delta,k}(r,f^{(1)}) \geq m_{\delta,k}(r,f)\left[\frac{\log m_{\delta,k}(r,f)}{\delta r \log r}\right]^{\delta}. \tag{2.3}$$

Also if $f(z)$ is an integral function of order $\varrho > 1$, then

$$\begin{aligned} m_{\delta,k}(r,f^{(1)}) &= \frac{1}{\pi r^{k+1}}\int_0^r\int_0^{2\pi} |f^{(1)}(xe^{i\theta})|^\delta x^k dx d\theta \\ &= \frac{1}{\pi r^{k+1}}\int_0^r\int_0^{2\pi}\left|\frac{f^{(1)}(xe^{i\theta})}{f(xe^{i\theta})}\right|^\delta |f(xe^{i\theta})|^\delta x^k dx d\theta \\ &< \frac{A}{\pi r^{k+1}}\int_0^r\int_0^{2\pi} x^{\delta(\rho+\epsilon-1)} |f(xe^{i\theta})|^\delta x^k dx d\theta \\ &< \frac{A}{\pi r^{k+1}} r^{\delta(\rho+\epsilon-1)}\int_0^r\int_0^{2\pi} |f(xe^{i\theta})|^\delta x^k dx d\theta, \quad \delta > 1, \\ &= A r^{\delta(\rho+\epsilon-1)} m_{\delta,k}(r,f). \end{aligned}$$

on using the result [Srivastav, 1957, p. 363], $\left|\frac{f^{(1)}(re^{i\theta})}{f(re^{i\theta})}\right| \leq O(r^{(\rho+\epsilon-1)})$.

Hence,

$$m_{\delta,k}(r, f^{(1)}) < A r^{\delta(\rho+\epsilon-1)} m_{\delta,k}(r,f), \tag{2.4}$$

where A is independent of r.

From (2.3) and (2.4) follows

$$\limsup_{r\to\infty} \frac{\log\left[r\left\{\frac{m_{\delta,k}(r,f^{(1)})}{m_{\delta,k}(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} = \varrho.$$

3. **Theorem 2.** *For every entire function $f(z)$, other than a polynomial,*

$$\limsup_{r\to\infty} \frac{\log\left[r\left\{\frac{\mu_\delta(r,f^{(1)})}{\mu_\delta(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} = \varrho, \quad 0 < \delta < 1,$$

where r tends to infinity through values outside an exceptional set of at most finite measure.

The proof of this theorem is based on the following two lemmas:

Lemma 1. *For every entire function $f(z)$, outside an exceptional set of at most finite measure* [Valiron, 1923, p. 103],

$$\mu_\delta(r,f^{(1)}) \geq \mu_\delta(r,f)\left(\frac{\nu(r)}{r}\right)^\delta\left\{1 - k\nu(R)^{-\frac{1}{16}}\right\}^\delta,$$

where $\nu(r)$ denotes the rank of the maximum term in $f(z)$, for $|z| = r$ and $0 < \delta < 1$.

Proof. We have
$$\mu_\delta(r,f^{(1)}) = \frac{1}{2\pi}\int_0^{2\pi} |f^{(1)}(re^{i\theta})|^\delta d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f^{(1)}(re^{i\theta})}{f(re^{i\theta})}\right| |f(re^{i\theta})|^\delta d\theta.$$

For ordinary values of r, we have [Valiron, 1923, p. 103]
$$\frac{f^{(1)}(z)}{f(z)} = [1+h(z)\nu(R)^{-\frac{1}{16}}]\frac{\nu(r)}{z}, \quad |h| < k, \quad |z| = r.$$

Hence,
$$\mu_\delta(r,f^{(1)}) = \frac{1}{2\pi}\int_0^{2\pi} |\{1+h(z)\nu(R)^{-\frac{1}{16}}\}|^\delta\left(\frac{\nu(r)}{r}\right)^\delta |f(re^{i\theta})|^\delta d\theta$$
$$\geq \left(\frac{\nu(r)}{r}\right)^\delta \frac{1}{2\pi}\{1-k\nu(R)^{-\frac{1}{16}}\}^\delta \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta,$$

i.e.
$$\mu_\delta(r,f^{(1)}) \geq \mu_\delta(r,f)\left(\frac{\nu(r)}{r}\right)^\delta \{1-k\nu(R)^{-\frac{1}{16}}\}^\delta.$$

Lemma 2. *For $0 < \delta < 1$ and for every positive value of ϵ,*
$$\mu_\delta(r,f^{(1)}) < A\mu_\delta(r,f)r^{(\rho+\epsilon-1)\delta},$$
where A is independent of r.

Proof. We have
$$\mu_\delta(r,f^{(1)}) = \frac{1}{2\pi}\int_0^{2\pi} |f^{(1)}(re^{i\theta})|^\delta d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f^{(1)}(re^{i\theta})}{f(re^{i\theta})}\right|^\delta |f(re^{i\theta})|^\delta d\theta$$
$$< \frac{A}{2\pi}r^{(\rho+\epsilon-1)\delta}\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta$$
$$= A\mu_\delta(r,f)r^{(\rho+\epsilon-1)\delta}$$

on using the result,
$$\left|\frac{f^{(1)}(re^{i\theta})}{f(re^{i\theta})}\right| \leq O(r^{(\rho+\epsilon-1)}).$$

Hence,
$$\mu_\delta(r,f^{(1)}) < A\mu_\delta(r,f)r^{(\rho+\epsilon-1)\delta}.$$

Proof of Theorem 2. From lemma 1, follows
$$\limsup_{r\to\infty} \frac{\log\left[r\left\{\frac{\mu_\delta(r,f^{(1)})}{\mu_\delta(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} \geq \limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \rho$$

Also from lemma 2, follows

$$\limsup_{r\to\infty} \frac{\log\left[r\left\{\frac{\mu_\delta(r,f^{(1)})}{\mu_\delta(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} \leq \varrho$$

Hence,

$$\limsup_{r\to\infty} \frac{\log\left[r\left\{\frac{\mu_\delta(r,f^{(1)})}{\mu_\delta(r,f)}\right\}^{\frac{1}{\delta}}\right]}{\log r} = \varrho.$$

4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an integral function of order ϱ and lower order λ, $M(r)$ the maximum modulus of $f(z)$ for $|z| = r$. Further it is known [Rahman (1956), p. 192] that if $f(z)$ is regular in $|z| \leq R$, and if $Z = re^{i\theta}$, $0 \leq r < R$, $\delta > 0$,

$$|f(z)|^\delta \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,|f(Re^{i\varphi})|^\delta d\varphi}{R^2-2Rr\cos(\theta)-\varphi+r^2}. \tag{4.1}$$

From (1.1) and (4.1) it is evident that

$$\mu_\delta(r) \leq \{M(r)\}^\delta \leq \frac{R+r}{R-r}\,\mu_\delta(R). \tag{4.2}$$

Theorem 3. *If $\nu(r)$ denotes the rank of the maximum term, $\mu(r)$, in the Taylor expansion of $f(z) = \sum_{n=0}^{\infty} a^n z^n$, $|z| = r$ and $0 < \lambda < \varrho \leq \infty$, then*

$$\limsup_{r\to\infty} \frac{\log m_{\delta,k}(r)}{\nu(r)\log r} \leq \delta\left(1-\frac{\lambda}{\varrho}\right),$$

$$\limsup_{r\to\infty} \frac{\log m_{\delta,k}(r)}{\nu(r)\log \nu(r)} \leq \delta\left(\frac{1}{\lambda}-\frac{1}{\varrho}\right),$$

and if $f(z)$ be of regular growth, then

$$\lim_{r\to\infty} \frac{\log m_{\delta,k}(r)}{\nu(r)\log r} = 0,$$

$$\lim_{r\to\infty} \frac{\log m_{\delta,k}(r)}{\nu(r)\log \nu(r)} = 0.$$

Proof. For functions of finite order we know that [Srivastav (1956), p. 80-81]

$$\limsup_{r\to\infty} \frac{\log M(r)}{\nu(r)\log r} = \limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r)\log r} \leq \left(1-\frac{\lambda}{\varrho}\right); \tag{4.3}$$

$$\limsup_{r\to\infty} \frac{\log M(r)}{\nu(r)\log \nu(r)} = \limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r)\log \nu(r)} \leq \left(\frac{1}{\lambda}-\frac{1}{\varrho}\right). \tag{4.4}$$

Now, using the inequality (4.2) for $|z| = x \leq r$, $k > 0$, we get

$$\mu_\delta(x) \leq \{M(x)\}^\delta,$$

or

$$\frac{2}{r^{k+1}}\int_0^r \mu_\delta(x)x^k dx \leq \frac{2}{r^{k+1}}\int_0^r \{M(x)\}^\delta x^k dx$$

$$\leq \frac{2}{r^{k+1}}\{M(r)\}^\delta \int_0^r x^k dx$$

$$= \frac{2}{k+1}\{M(r)\}^\delta,$$

i.e.

$$m_{\delta,k}(r) \leq \frac{2}{k+1}\{M(r)\}^\delta.$$

Since $\nu(r)$ is positive and unbounded function of r, therefore, taking limits dividing by $\nu(r)\log r$, we get

$$\lim_{r\to\infty} \text{up} \frac{\log m_{\delta,k}(r)}{\nu(r)\log r} \leq \lim_{r\to\infty} \sup \frac{\delta \log M(r)}{\nu(r)\log r} = \lim_{r\to\infty} \sup \frac{\delta \log \mu(r)}{\nu(r)\log r}.$$

Hence from (4.3), $$\lim_{r\to\infty} \sup \frac{\log m_{\delta,k}(r)}{\nu(r)\log r} \leq \delta\left(1 - \frac{\lambda}{\varrho}\right).$$

Again, $$\lim_{r\to\infty} \sup \frac{\log m_{\delta,k}(r)}{\nu(r)\log \nu(r)} \leq \lim_{r\to\infty} \sup \frac{\delta \log M(r)}{\nu(r)\log \nu(r)} = \lim_{r\to\infty} \sup \frac{\delta \log \mu(r)}{\nu(r)\log \nu(r)}.$$

Therefore, from (4.4),

$$\lim_{r\to\infty} \sup \frac{\log m_{\delta,k}(r)}{\nu(r)\log \nu(r)} = \delta\left(\frac{1}{\lambda} - \frac{1}{\varrho}\right).$$

5. Theorem 4. *Let $f(z)$ be an integral function, other than a polynomial, of finite order ϱ, $n(r)$ denote the number of zeros of $f(z)$ in $|z| \leq r$ and $f(0) \neq 0$. Further,*

(i) *if*
$$\lim_{r\to\infty} \inf \frac{n(r)}{r^\rho} = \beta,$$

then
$$\lim_{r\to\infty} \inf \frac{\log m_{\delta,k}(r)}{r^\rho} \geq \frac{\beta\delta}{\varrho(\varrho+1)}; \tag{5.1}$$

(ii) *if*
$$\lim_{r\to\infty} \inf \frac{n(r)}{r\log r} > 1,$$

then
$$\lim_{r\to\infty} \inf \frac{\log m_{\delta,k}(r)}{r\log r} > \frac{\delta}{2}, \quad \delta > 0. \tag{5.2}$$

Proof. (*i*) We have

$$\log \mu_\delta(x) = \log\left[\frac{1}{2\pi}\int_0^{2\pi} |f(xe^{i\theta})|^\delta d\theta\right]$$

and since $|f(xe^{i\theta})|^\delta$ is a positive continuous function, we get from [Titchmarsh, p. 311]

$$\log \mu_\delta(x) \geq \frac{1}{2\pi}\delta \int_0^{2\pi} \log |f(xe^{i\theta})| \, d\theta$$

$$= \delta \int_0^x t^{-1}n(t)dt + \delta \log |f(0)|, \quad x \leq r, \tag{5.3}$$

on using Jenson's formula.

Hence, if $\liminf\limits_{r\to\infty} \dfrac{n(r)}{r^\rho} = \beta$, we have for any $\varepsilon > 0$ and $r > r_0$, $n(r) > (\beta-\varepsilon)r^\rho$, therefore, $\log \mu_\delta(x) > \delta(\beta-\varepsilon)x^\rho/\varrho + \delta \log |f(0)|$, for $x > r_0$.

Since $\log\{\mu_\delta(x)x^k\}$ is a positive function,

$$\frac{1}{r}\int_0^r \log\{\mu_\delta(x)x^k\}dx \geq \frac{1}{r}\int_{r_0}^r \log\{\mu_\delta(x)x^k\}dx > \frac{(\beta-\varepsilon)\delta}{\varrho(\varrho+1)}\left(r^\rho - \frac{r_0^{\rho+1}}{r}\right)$$

$$+\delta \log |f(0)|.\left(1-\frac{r_0}{r}\right) + k\left\{\left(\log r - \frac{r_0}{r}\log r_0\right) - \left(1-\frac{r_0}{r}\right)\right\}.$$

Since

$$\log\left\{\frac{1}{r}\int_0^r \mu_\delta(x)x^k dx\right\} \geq \frac{1}{r}\int_0^r \log\{\mu_\delta(x)x^k\}dx,$$

therefore,

$$\log\left\{\frac{1}{r}\int_0^r \mu_\delta(x)x^k dx\right\} > \frac{(\beta-\varepsilon)\delta}{\varrho(\varrho+1)}\left(r^\rho - \frac{r_0^{\rho+1}}{r}\right) + \delta\log|f(0)|.\left(1-\frac{r_0}{r}\right)$$

$$+k\left\{\left(\log r - \frac{r_0}{r}\log r_0\right) - \left(1-\frac{r_0}{r}\right)\right\}$$

or

$$\log m_{\delta,k}(r,f) > \frac{(\beta-\varepsilon)\delta}{\varrho(\varrho+1)} r^\rho + O(1)$$

Proceeding to limits, we get

$$\liminf_{r\to\infty} \frac{\log m_{\delta,k}(r,f)}{r^\rho} \geq \frac{\beta\delta}{\varrho(\varrho+1)}.$$

(*ii*) Again, if $\lim\limits_{r\to\infty} \dfrac{n(r)}{r\log r} > 1$, we have for any $\varepsilon > 0$ and $r > r_0$, $n(r) > (1-\varepsilon)r\log r$, and from (5.3), we get

$$\log \mu_\delta(x) > \delta(1-\varepsilon)\int_0^x \log t\,dt + \delta\log|f(0)|$$

$$= \delta(1-\varepsilon)(x\log x - x) + \delta\log|f(0)|,$$

or $\log\{\mu_\sigma(x)x^k\} > \delta(1-\varepsilon)(x\log x - x) + \delta\log|f(0)| + \log x^k$, for $r \geq x > r_0$.

therefore $$\frac{1}{r}\int_0^r \log\{\mu_\delta(x)x^k\}dx \geq \frac{1}{r}\int_{r_0}^r \log\{\mu_\delta(x)x^k\}dx$$

$$> \delta(1-\varepsilon)\left\{\tfrac{1}{2}\left(r\log r - \frac{r_0^2}{r}\log r_0\right) - \tfrac{1}{4}\left(r - \frac{r_0^2}{r}\right) - \tfrac{1}{2}\left(r - \frac{r_0^2}{r}\right)\right\} + \delta \log|f(0)|\left(1 - \frac{r_0}{r}\right)$$

$$+ k\left\{\left(\log r - \frac{r_0}{r}\log r_0\right) - \left(1 - \frac{r_0}{r}\right)\right\}.$$

Since $$\log\left\{\frac{1}{r}\int_0^r \mu_\delta(x)x^k dx\right\} \geq \frac{1}{r}\int_0^r \log\{\mu_\delta(x)x^k\}dx,$$

therefore, $$\log\left\{\frac{1}{r}\int_0^r \mu_\delta(x)x^k dx\right\} > \delta(1-\varepsilon)\left\{\tfrac{1}{2}\left(r\log r - \frac{r_0^2}{r}\log r_0\right) - \tfrac{3}{4}\left(r - \frac{r_0^2}{r}\right)\right\}$$

$$+ \delta \log|f(0)|.\left(1 - \frac{r_0}{r}\right) + k\left\{\left(\log r - \frac{r_0}{r}\log r_0\right) - \left(1 - \frac{r_0}{r}\right)\right\}.$$

Hence, $$\log m_{\delta,k}(r, f) > \tfrac{1}{2}\delta(1-\varepsilon)\{r\log r - \tfrac{3}{2}r\} + O(1).$$

Proceeding to limits, we get

$$\liminf_{r\to\infty} \frac{\log m_{\delta,k}(r, f)}{r\log r} > \frac{\delta}{2}.$$

6. Application. If $\lambda = 1$ and $\lim\limits_{r\to\infty} \dfrac{n(r)}{r\log r} > 2$, where $n(r)$ denotes the number of zeros of $f(z)$ in $|z| \leq r$, then $m_{\delta,k}(r, f) < m_{\delta,k}(r, f^{(1)}) < \ldots\ldots < m_{\delta,k}(r, f^{(n)}) < \ldots\ldots$ for all $r > r_0 = r_0(f)$.

From (5.2), we have $$\liminf_{r\to\infty} \frac{\log m_{\delta,k}(r,f)}{r\log r} > \delta.$$

Hence, from (2.3) with $\delta > 1$, for $r > r_0$,

$$\left\{\frac{m_{\delta,k}(r, f^{(1)})}{m_{\delta,k}(r, f)}\right\}^{\frac{1}{r}} \geq \left\{\frac{\log m_{\delta,k}(r, f)}{\delta r\log r}\right\} > 1.$$

i.e. $$m_{\delta,k}(r, f) < m_{\delta,k}((r, f^{(1)}).$$

Similar inequalities for higher derivatives give the result. I am grateful to Dr. S. K. Bose for suggesting the problem and his guidance in the preparation of this paper.

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GEOMETRICAL SIGNIFICANCE OF THE ELECTRO-MAGNETIC ENERGY-MOMENTUM TENSOR

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1. Introduction. In general relativity gravitation manifests itself as a geometrical property of the four-dimensional space-time continuum. It is described by a Riemannian metric satisfying Einstein's field equations

$$-8\pi T_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \tag{1}$$

which connect the distribution of matter and energy with the fundamental tensor $g_{\mu\nu}$ and its derivatives. The various forms of the energy-momentum tensor $T_{\mu\nu}$ such as those for perfect fluid or electro-magnetic distributions impose certain restrictions on the possible geometry of space-time continuum. A geometrico algebraic characterization of the space-time continua of perfect fluids and the electromagnetic energy-momentum tensor has been studied in detail by Eisenhart (1924) and Lichnerowicz (1955) respectively. However, a global aspect of the space-time characterized by these distributions is, sometimes, well-exhibited in terms of the Euclidean spaces of higher dimensions in which these can be immersed. If the minimum number of dimensions of a Euclidean space in which a given Riemannian space of n-dimensions can be immersed is $n+p$, the latter is said to be of class p. Karmarkar (1948) has studied some spherically symmetric metrics and by imposing class one restriction he has obtained Schwarzschild's internal solution from which the cosmological models of Einstein and de Sitter follow as particular cases. In a recent attempt towards a classwise study of Riemannian fourfolds, Singh and Pandey (1960) have been able to deduce a new form of non-static line-element for Lemaitre's universe from class one considerations.

While all these investigations show that a perfect fluid distribution of matter is compatible with class one metrics, it is found in the present investigation that the electromagnetic energy-momentum tensor as a source term in Einstein's gravitational field equations is not compatible with Riemannian fourfolds of class one insofar as spherical symmetry is concerned.

2. Electro-magnetic energy-momentum tensor and class one. A necessary and sufficient condition for a spherically symmetric metric

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu} dt^2, \quad \lambda = \lambda(r,t), \quad \nu = \nu(r,t), \tag{2}$$

to be of class one has been obtained by Karmarkar in the form

$$8(1-e^{-\lambda})^2+8\pi r^2(1-e^{-\lambda})\left(4\overset{2}{T}_2-\overset{1}{T}_1-\overset{4}{T}_4\right)-64\pi r^4\left(\overset{1}{T}_1\overset{4}{T}_4-\overset{4}{T}_1\overset{1}{T}_4\right)=0 \tag{3}$$

where $T_\mu^{\;\nu}$ are the components of the energy-momentum tensor defined by (1). We proceed to examine the compatibility of the electro-magnetic energy tensor (Eddington, 1923).

$$T_\mu^{\;\nu}=-F^{\nu\alpha}F_{\mu\nu}+\tfrac{1}{4}g_\mu^{\;\nu}F^{\alpha\beta}F_{\alpha\beta} \tag{4}$$

with (1), (2) and (3).

It is well-known that $\overset{1}{T}_2$, $\overset{1}{T}_3$, $\overset{2}{T}_3$, $\overset{2}{T}_4$, $\overset{3}{T}_4$ all vanish and $\overset{2}{T}_2=\overset{3}{T}_3$ in case of (2). When these conditions are imposed on $T_\mu^{\;\nu}$ defined in (4) we get as a result the relations

$$F^{12}F_{23}+F^{14}F_{24}=0 \tag{5}$$

$$F^{13}F_{14}+F^{23}F_{21}=0 \tag{6}$$

$$F^{12}F_{23}+F^{14}F_{43}=0 \tag{7}$$

$$F^{12}F_{13}+F^{24}F_{34}=0 \tag{8}$$

$$F^{12}F_{14}+F^{23}F_{43}=0 \tag{9}$$

$$F^{12}F_{12}+F^{24}F_{24}=F^{13}F_{13}+F^{34}F_{34}. \tag{10}$$

Equations (5) – (10) imply that

$$F_{23}=0,\quad F_{14}=0,\quad F_{12}=e^{\lambda-\nu/2}F_{34},\quad F_{13}=e^{\lambda-\nu/2}F_{34}, \tag{11}$$

which signify that all the components of $F_{\alpha\beta}$ become known if F_{12} and F_{13} are known. From (4) and (11) it can be verified that

$$\overset{2}{T}_2=\overset{3}{T}_3=0, \tag{12}$$

$$\overset{1}{T}_1+\overset{4}{T}_4=0, \tag{13}$$

$$\overset{1}{T}_1\overset{4}{T}_4-\overset{1}{T}_4\overset{4}{T}_1=0. \tag{14}$$

When use is made of (12) – (14) in (3), the equation for λ turns out to be

$$e^\lambda=1. \tag{15}$$

Substituting from (15) in the expressions for $T_\mu^{\;\nu}$ given by Tolman (1934a) and making use of (12) and (13) the following two equations for ν are obtained :

$$\nu'/r+2\Lambda=0, \tag{16}$$

$$2\Lambda+\nu''+\nu'^2/2+\nu'/r=0, \tag{17}$$

where an overhead dash indicates a partial differentiation with respect to r. Consistency of (16) and (17) demands that Λ and ν' should vanish together. Hence ν is a function of t alone and the line-element (2) reduces to a flat metric. Thus it follows that a spherically symmetric line-element of class one cannot describe an electro-magnetic situation.

It may be pointed ont that some of the well-known solutions, obtained by the use of Maxwell's stress-energy tensor as source term in Einstein's gravitational field equations, are in agreement with this class property. For example one may mention here that Nordström's (1918) solution describing the gravitational field of a charged particle is a static spherically symmetric metric of class two. Narlikar and Vaidya (1947) have obtained a non-static spherically symmetric line-element describing the gravitational field due to a directed flow of electro-magnetic radiation. The author has verified that the metric so obtained is of class two.

It is important to note here that a disordered distribution of radiation which Tolman (1934b) defines as a special case of perfect fluid distribution characterised by the relation $\varrho = 3p$ is not incompatible with class one conditions. However, a confusion is created by the use of the terminology 'a disordered distribution of electromagnetic energy' insofar as the above established class property of electromagnetic distribution is concerned. That the case $\varrho = 3p$ may be described by a metric of class one does not contradict the above result. This is clear from the fact that the stress-energy system in this case is not Maxwellian.

While discussing the geometrical signifincance of the electromagnetic energy tensor one may also recall a result in unified field theory. In an investigation of the distant parallelism theory of Einstein, later modified by Levi-Civita, Tiwari (1951) has shown that the electromagnetic field totally disappears for a metric of the form

$$ds^2 = -Adx^2 - Bdy^2 - Cdz^2 + Ddt^2$$

where A, B, C and D may be functions of the coordinates.

The author is grateful to Prof. V. V. Narlikar for suggesting the above investigation. His thanks are also due to Dr. K. P. Singh for many useful discussions.

Summary. It is shown that the electromagnetic energy-momentum tensor as a source term in Einstein's gravitational field equations is not compatible with spherically symmetric line-elements of class one.

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ON SETS UNDER CERTAIN TRANSFORMATIONS—II

By

B. K. Lahiri, *Calcutta*

(*Received—October 17, 1960*)

Kestelman (1947) has proved a number of interesting results in a bounded closed set of positive measure. He has proved among other results that if C is a bounded closed set of positive measure in E_n (n dimensional Euclidean space) and p be any positive integer, then there exists a positive number δ (depending on p and the measure of C) such that if $\lambda_1, \lambda_2, \dots \lambda_p$ are any p vecors in E_n whose lengths $|\lambda_r|$ satisfy the relations $|\lambda_r| < \delta$, $r = 1, 2, \dots p$; then it is always possible to find a closed set $C_1 \subset C$ of positive measure having the property that if ξ is in C_1, then $\xi + \lambda_r$ is in C, $r = 1, 2, \dots p$.

Now, if $T_r (r = 1, 2, \dots, p)$ denote the transformation of translation

$$\text{viz.} \qquad x_i' = x_i + a_i^r, \quad i = 1, 2, \dots n; \quad a^r \text{ real},$$

then we see from the result of Kestelman that he has, in fact, proved that if C is any bounded closed set of positive measure in E_n and p be any positive integer, then there exists a positive number δ (depending on p and the measure of C) such that if

$$\lambda_r = (a_1^r, a_2^r, \dots a_n^r), \qquad r = 1, 2, \dots, p$$

are vectors in E_n satisfying $|\lambda_r| < \delta$, then the set

$$C(T_1 C)(T_2 C) \dots (T_p C) = X \text{ (say)}$$

is a closed set of positive measure.

(By $T_r C$ we mean the set of points which are points of E_n to which the points of C are transformed under T_r). In a previous paper (Lahiri, 1959a) we have proved a number of results similar to those of Kestelman by considering the transformation T_r in the following manner

$$T_r : \; x_i' = b_r x_i, \qquad i = 1, 2, \dots n; \; b_r \text{ real}$$

where $b_r \neq 0$ satisfy certain conditions. It has been shown there that under some restrictions on b_r, the set X as defined above forms a closed set of positive measure.

In a later paper (Lahiri, 1959b) we enquire whether the set X as defined above forms a closed set of positive measure if T_r $(r = 1, 2, \dots p)$ be any general linear transformation of the form

$$x_i' = \sum_{j=1}^{n} a_{ij}^r x_j + a_{i,n+1}^r \quad i = 1, 2, \dots n;\ a_{ij}^r \text{ real},$$

where we assume that the determinant of the transformation, viz.

$$\begin{vmatrix} a_{11}^r & \dots & a_{1n}^r \\ \dots & & \dots \\ a_{n1}^r & \dots & a_{nn}^r \end{vmatrix}$$

is different from zero and a_{ij}^r satisfy certain suitable conditions.

Here also we have obtained some results parallel to those of Kestelman.

In this paper we give a geometrical (functional) interpretation of the generalization of some of the results of Kestelman. Here the transformations are treated as functions and a neighbourhood is defined for each function making the aggregate of such functions into a topological space and the transformations (functions) are treated as points of this space.

We proceed as follows :

Let T denote the linear nonsingular transformation, viz.

$$x_i' = \sum_{j=1}^{n} a_{ij} x_j + a_{i,n+1},\ i = 1, 2, \dots n; \quad a_{ij} \text{ real}.$$

The transformation T is completely determined by the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{nn+1} \end{pmatrix}$$

So, if we are supplied with the matrix of the transformation only, then we can fully understand the nature of the transformation and obviously different matrices give rise to different transformations. Now, if we write.

$$f = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{nn+1} \end{pmatrix}$$

then by the function f we shall understand the correspondence between a set $C(\subset E_n)$ and the set of points of E_n which are the transformed points of C under the transformation T. Since T is nonsingular f transforms E_n onto E_n in a bi-uniform and bi-continuous manner. We denote by $f(C)$ the transform of $C \subset E_n$ by the function f (i.e. transformation T)

Let now ϵ be any positive number and

$$g = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} & x_{1n+1} \\ x_{21} & x_{22} & \cdots & x_{2n} & x_{2n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} & x_{nn+1} \end{pmatrix}$$

where x_{ij} be any real number such that

$$a_{ij}-\epsilon < x_{ij} < a_{ij}+\epsilon, \; i = 1, 2, \ldots, n \; ; \quad j = 1, 2, \ldots n+1.$$

Then we say that the set of functions g constitutes a neighbourhood of f. If the numbers x_{ij} be such that $a_{ij} \leqslant x_{ij} < a_{ij}+\epsilon$, then we say that the set of functions g constitutes a right neighbourhood of f. Similar is the case for left neighbourhood. Having given $\epsilon > 0$, we say that the neighbourhood of f is determined by ϵ.

We present our main result of the note as follows :

Let C be a bounded closed set of positive measure contained in E_n and let I denote the function determined by the $n \times (n+1)$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

which leaves every point of E_n invariant.

Then corresponding to any positive integer p and positive number δ (depending on p and the measure of C), we can determine a right neighbourhood of I such that if $f_1, f_2, \ldots f_p$ be any p functions lying in this neighbourhood then that set

$$C \,.\, f_1(C) \,.\, f_2(C) \; \ldots \; f_p(C)$$

forms a closed set of positive measure.

In this connection, we recapitulate a well-known result (Caratheodory, 1948). If C denote any set in E_n and T any linear transformation then

$$m^*(TC) = |D| \, m^*(C)$$

$$m_*(TC) = |D| \, m_*(C)$$

where $|D|$ is the absolute value of the determinant of the transformation and $m^*(X)$, $m_*(X)$ denote the Lebesgue exterior and interior measure of X. So if C is Lebesgue measurable then TC is also so and

$$m(TC) = |D|\, m(C).$$

We shall in the following understand by the term 'measure of a set' its Lebesgue measure.

Theorem 1. *Let C be a bounded closed set of positive measure contained in E_n and p be any positive integer. Then it is possible to find a positive number M and a number $\delta > 0$ depending on p and the measure of C such that if I is the function determined by the $n \times (n+1)$ matrix*

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and $f_1, f_2, \dots f_p$ be any p functions belonging in the right neighbourhood of I determined by the number $\delta/(M+1)n$, then the set of points belonging to

$$C \,.\, f_1(C) \,.\, f_2(C) \dots f_p(C)$$

forms a closed set of positive measure.

Proof. Since C is bounded, there exists a sphere of radius, say M containing C. Now, let U be an open set containing C and contained in the sphere of radius M satisfying the relation

$$m(U) - m(C) < m(C) . \frac{2p-1}{3p(p+1)} \tag{1}$$

Now, if $\delta > 0$ be the distance between C and U' (complement of U) then we may easily choose the set U in such a way that

$$\delta < \frac{1}{3p} \cdot \frac{1}{2^n n!} \tag{2}$$

Now, a point ξ of C corresponds uniquely to a point ξ_r' of $f_r(C)$ by the function f_r $(r = 1, 2, \dots p)$. Since C is closed and f_r is continuous, so $f_r(C)$ is closed.

The function $f_r (r = 1, 2, \dots, p)$ is determined by the matrix

$$\begin{pmatrix} b_{11}^r & b_{12}^r & \dots & b_{1n}^r & b_{1n+1}^r \\ \dots & \dots & \dots & \dots & \dots \\ b_{n1}^r & b_{n2}^r & \dots & b_{nn}^r & b_{nn+1}^r \end{pmatrix}$$

where b^r_{ij} has values such that f_r lies in the right neighbourhood of I determined by the number $\delta/(M+1)n$. So, the determinant of the transformation which gives rise to the correspondence between C and $f_r\ (C)$ is

$$\begin{vmatrix} b^r_{1} & \dots & b^r_{1n} \\ \dots & \dots & \dots \\ b^r_{n1} & \dots & b^r_{nn} \end{vmatrix} = D_r \quad \text{(say)}$$

Therefore the measure of $f_r\ (C)$ is given by

$$m(f_r(C)) = |D_r|\, m(C).$$

Now, if X' is the corresponding point of a point X of C under f_r then it can be easily verified from the conditions on the elements b^r_{ij} characterizing the function f_r that

$$|X'-X| < \delta \quad (|x-y| \text{ denotes the distance between } x \text{ and } y)$$

So, $$f_r(C) \subset U,\ r = 1, 2, \dots p$$

Again, it is easily obtained that under the conditions on the elements b^r_{ij}

$$D_r > 1 - \Sigma\, b_{i_r j_r}\, b_{i_s j_s} \dots b_{i_t j_t},$$

where the summation contains $n!/2$ terms.

Again, it can be easily seen that

$$a_{i_r j_r}\, a_{i_s j_s} \dots a_{j_t i_t} < \left(1 + \frac{\delta}{(M+1)n}\right)^{n-2} \cdot \frac{\delta}{(M+1)n} \cdot \frac{\delta}{(M+1)n}.$$

So, $$D_r > 1 - \frac{n!}{2}\left(1 + \frac{\delta}{(M+1)n}\right)^{n-2} \cdot \frac{\delta^2}{(M+1)^2 n^2}$$

$$> 1 - 2^n \,.\, n! \,.\, \delta$$

$$> 1 - \frac{1}{3p} \quad \text{from (2).}$$

So, $$m(f_r(C) = D_r m(C) > \left(1 - \frac{1}{3p}\right) m(C), \text{ for } r = 1, 2, \dots p.$$

Let Z denote the set $f_0(C) \,.\, f_1(C) \,.\, f_2(C) \,.\, \ldots f_p\ (C)$, $f_0(C) = C$.

Then
$$Z = \prod_{r=0}^{p} f_r(C) = U - \sum_{r=0}^{p} \{U - f_r(C)\}$$

$$\therefore \quad m(Z) \geqslant m(U) - \sum_{r=0}^{p} \{m(U) - m(f_r(C))\} > m(U) - (p+1)\left\{m(U) - \left(1 - \frac{1}{3p}\right) m(C)\right\}$$

From (1)
$$m(U) - m(C)\left(1 - \frac{1}{3p}\right) < \frac{m(C)}{p+1}$$

So,
$$m(Z) > m(U) - (p+1)\frac{m(C)}{p+1} > 0$$

This proves the theorem.

Theorem. 2 *Let C be a closed set of positive measure contained in an open sphere of radius, say M in E_n. Let $\{\delta_r\}$ be any positive null sequence and f_r be any function lying in the right neighbourhood of I (I as defined in Theorem 1) determined by the number $\delta_r/(M+1)n$. Then there exists a subsequence $\{\delta_{n_r}\}$ of $\{\delta_r\}$ and a point X such that*

$$X \text{ is in } C \,.\, f_{n_1}(C) \,.\, f_{n_2}(C) \,.\, f_{n_3}(C) \ldots$$

This theorem can be easily proved.

Theorem 3. *Let C be a closed set of positive measure contained in an open sphere of radius, say M in E_n. Then we can find a positive null sequence $\{\mu_r\}$ such that if $\{\delta_r\}$ be any null sequence satisfying*

$$0 < \delta_r < \min\,[\mu_r,\ 2^{n+r+1}.\ n!]$$

and $f_{\delta r}$ is any function lying in the right neighbourhood of I (I as in Theorem 1) determined by the number $\delta_r/(M+1)n$, then the set of points X belonging to

$$C \,.\, f_{\delta_1}(C) \,.\, f_{\delta_2}(C) \,.\, f_{\delta_3}(C) \ldots$$

forms a closed set of positive measure.

Proof As shown in Kestelman's paper we can obtain a sequence of open sets $\{U_r\}$ such that

$$U_1 \supset U_2 \supset U_3 \supset \ldots; \quad C = \prod_{r=1}^{\infty} U_r$$

and
$$\sum_{r=1}^{\infty} \{m(U_r) - m(C)\} < \frac{m(C)}{2}.$$

If $\mu_r\,(>0)$ is the distance between C and U_r' (complement of U_r) then evidently $\{\mu_r\}$ is a positive null sequence. We now choose the number $\delta_r > 0$ such that

$$\delta_r = \min\left[\mu_r, \frac{1}{2^{n+r+1}n!}\right]$$

Then $\{\delta_r\}$ is also a positive null sequence. Let f_{δ_r} by any function lying in the right neighbourhood of I determined by the number $\delta_r/(M+1)n$. Now, if X' is the corresponding point of a point X of C under f_{δ_r} then it can be easily shown from the condition on the elements characterizing the function f_{δ_r} that $|X - X'| < \delta_r$.

So, $f_{\delta_i}(C) \subset U_r$ for every r.

Let $Z = f_{\delta_1}(C).f_{\delta_2}(C).f_{\delta_3}(C)\ldots$

then if ξ is in Z, ξ is in $f_{\delta_n}(C)$ for all n, i.e. dist $(\xi, C) < \delta_n$ for all n.

Since C is closed, ξ is in C.

Again, since each $f_{\delta_r}(C)$ is closed, so Z is closed. So, we are only to show that the measure of Z is positive.

Now,
$$Z = f_{\delta_1}(C).f_{\delta_2}(C).f_{\delta_3}(C)\ldots$$

$$= U_1 - \sum_{r=1}^{\infty} \{U_1 - f_{\delta_r}(C)\}$$

$$= U_1 - \Sigma\{U_1 - U_r + U_r - f_{\delta_r}(C)\}$$

$$= C - \Sigma\{U_r - f_{\delta_r}(C)\}$$

$$\therefore \quad m(Z) \geqslant m(C) - \Sigma\{m(U_r) - m(f_{\delta_r}(C))\}$$

As in Theorem 1, we see that

$$m(f\delta_r(C)) > (1-2^n.\,n!.\,\delta_r)\, m(C) > m(C) - \frac{m(C)}{2^{r+1}}\,.$$

$$\therefore\quad m(U_r) - m(f\delta_r(C)) < m(U_r) - m(C) + \frac{m(C)}{2^{r+1}}$$

$$\therefore\quad m(Z) > m(C) - \Sigma\left\{m(U_r) - m(C) + \frac{m(C)}{2^{r+1}}\right\}$$

$$= \frac{m(C)}{2} - \Sigma\{m(U_r) - m(C)\} > 0.$$

This proves the theorem.

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SOME ASPECTS OF SUPERPOSABILITY AND SELF-SUPERPOSABILITY OF FLUID MOTIONS IN GASDYNAMICS

By

J. N. Kapur, *Kanpur*

(*Received—September 24, 1960*)

1. Introduction. To deal with the non-linear character of the hydrodynamical equations of motion, Ram Ballabh (1940a) introduced the concept of superposability and self-superposability in 1940. After that the subject was investigated by a number of authors Ram Ballabh [1–6] Bhatnagar (1957), Ergun (1954, 1944), Kapur (1959a, 1960a, 1960b), Ghildyal (1957, 1954), Prem Prakash (1954, 1955) and Strang (1942, 1948), all of whom confined themselves to the case of an incompossible viscous Newtonian fluid. Gold and Krzywoblocki (1958a) in a very comprehensive study of the subject extended this concept to the cases of (i) incompressible non-Newtonian fluids (ii) compressible Newtonian fluids (iii) Compressible non-Newtonian fluids (iv) viscous conducting fluids under the action of magnetic fields. Independently of this Srivastava (1956a, 1956b) studied superposability in Non-Newtonian fluids and Kapur (1959b, 1960c), Bhatnagar (1958-59), Ram Moorty (1960), and Teeka Rao (1959, 1960) investigated superposability in magneto-hydrodynamics. Kapur (1959b, 1960c) also gave a number of applications of this concept to the case of viscous conducting fluids.

For steady irrotational motion of a non-viscous, non-heat-conducting gas, the equations of motion and continuity can be combined into the gasdynamics potential equation:

$$\begin{aligned}(a^2-\phi_x^2)\phi_{xx}+(a^2-\phi_y^2)\phi_{yy}+(a^2-\phi_z^2)\phi_{zz}\\ -2\phi_x\,\phi_y\,\phi_{xy}-2\phi_x\phi_z\phi_{xz}-2\phi_y\phi_z\phi_{yz}=0,\end{aligned} \tag{1}$$

where ϕ is the velocity potential and a is the velocity of sound given by

$$a^2 = a_0^2-\tfrac{1}{2}(\gamma-1)(\phi_x^2+\phi_y^2+\phi_z^2), \tag{2}$$

where a_0 is the stagnation sound velocity and γ is the ratio of specific heats. Gold and Krzywoblocki (1958a, 1958b) referring to (1) state that "in the given form, it is extremely difficult, if possible, to apply superposability in the established manner". They therefore applied a limiting-process to solve the potential equation by reducing the solution of this non-linear equation to the solution of an infinite number of linear second order partial differential equations provided that the series of the sum of the solutions of these equations is convergent. The convergence is however not easy to discuss.

In their discussion of the concept of superposability to viscous heat-conducting compressible fluid motions, Gold and Krzywoblocki (1958a, 1958b) have been obliged to treat the variable Reynold number as a parameter with apparently no other justifiaction except that of mathematical tractability. This approach results in overspecifying the system and at the same time the system loses all physical meaning.

There appears therefore the need for another approach to superposability in compressible fluids. The beginnings of such an approach have already been discussed by the author (Kapur 1960b) in another paper and this approach will be further discussed in the present paper where the results of the series of papers on steady rotational gas flows by Prim, Munk, Nemenyi Portisky and others (1945, 1948, 1944, 1958, 1958, 1944b, 1948a, 1948b 1948b, 1952) have been used. These results are stated to have been made possible by the reduction of the basic equations to the canonical form (1947) in the terms of the reduced velocity vector $\boldsymbol{\omega} = \boldsymbol{\nu}/a'$ where a' is the ultimate velocity along a stream-line and their approach was stated to be not practicable using the older formulation of the equations. It is possible however to obtain all these results and sometimes results of even greater generality without such a canonical reduction. Accordingly we have proved independently from first principles those results which we require in our investigation.

The main object of the present paper is to obtain three dimensional solutions from two-dimensional ones by using the principle of superposability. It is well-known that the number of exact analytical solutions for the basic equation of gasdynamics is very limited. Almost all of these—two dimensional vortex flow, radial flow, spiral flow, Prandtl-Meyer flow, flow past a wedge, Taylor-Maccoll flow past a cone and Ringlebs transonic flow—have been discussed in Howarth (1953), and Pai (1960). Again most of these are two-dimensional flows and we shall give a general method of constructing three-dimensional flows based on these two-dimensional flows. The number of exact solutions at our disposal will be considerably enlarged and these will be in terms of the velocity vector rather than in terms of the reduced velocity vector.

2. The Basic Equations for Steady Flow. The basic equations for frictionless steady flow of a compressible fluid, in tensor notation, are :

$$\frac{\partial}{\partial x_j}(\rho u_j) = 0 \qquad \text{[continuity]} \tag{2}$$

$$\rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} \qquad \text{[momentum]} \tag{3}$$

$$u_j \frac{\partial s}{\partial x_j} = 0 \qquad \text{[energy]} \tag{4}$$

$$\rho = P(p)S(s) \qquad \text{[state]} \tag{5}$$

where u_j are the velocity components and p, ρ, s denote pressure density and specific entropy respectively. Also we are using the 'product' equation of state which includes the equation of state for a perfect gas as a particular case.

From equation (5)

$$\frac{1}{\rho}\frac{\partial \rho}{\partial x_j} = \frac{P'(p)}{P(p)}\frac{\partial p}{\partial x_j} + \frac{S'(s)}{S(s)}\frac{\partial s}{\partial x_j}. \tag{6}$$

Multiplying both sides of (6) by u_j and substituting for

$$u_j\frac{\partial \rho}{\partial x_j},\quad u_j\frac{\partial p}{\partial x_j},\quad u_j\frac{\partial s}{\partial x_j} \text{ from (2), (3), (4) we get,}$$

$$\frac{\partial u_j}{\partial x_j} = \frac{P'(p)}{P(p)}\rho\, u_i u_j \frac{\partial u_i}{\partial x_j} \tag{7}$$

or

$$\frac{u_i u_j \dfrac{\partial u_i}{\partial x_j}}{\dfrac{\partial u_k}{\partial x_k}} = \frac{1}{\rho\dfrac{P'(p)}{P(p)}} \tag{7}$$

or

$$\rho = \frac{Y(p)}{X} \tag{8}$$

where $Y(p)$ and X are defined by (7) and (8). Substituting in (2) and making use of (3), we get

$$u_i u_j \frac{\partial u_i}{\partial x_j}\frac{d}{dp}(Y(p)) + u_j\frac{\partial X}{\partial x_j} - X\frac{\partial u_j}{\partial x_j} = 0 \tag{9}$$

This will be independent of the pressure if

$$Y(p) = \frac{P(p)}{P'(p)} = ap+b; \tag{10}$$

where a and b are constants.

Integrating (10), we get

$$P(p) = A(ap+b)^{1/a}, \tag{11}$$

where A is a constant.

Thus we find that pressure can be eliminated from the basic equations only if the equation of state is of the form*

$$P = A(ap+d)^{1/a}\, S(s) \tag{12}$$

* For an alternative proof of this fact see Prim (1952).

and then the equation, satisfied by the velocity components is given by,

$$(a-1) \quad u_i u_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \left(\frac{u_i u_k \dfrac{\partial u_i}{\partial x_k}}{\dfrac{\partial u_l}{\partial x_l}} \right) = 0 \tag{13}$$

This case, of course, includes the case when the gas is calorifically perfect, when the equation of state is given by

$$\rho = Bp^{\frac{1}{\gamma}} e^{-\frac{s-s_0}{c_p}}, \tag{14}$$

where B is a constant and $P(p)$ has been taken as $p^{\frac{1}{\gamma}}$

3. The Definitions of Superposaibility and Self-superposability for compressible Fluids. For a frictionless flow which is not necessarily steady, the basic equations are :

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \tag{2'}$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} \tag{3'}$$

$$\frac{\partial s}{\partial t} + u_j \frac{\partial s}{\partial x_j} = 0 \tag{4'}$$

$$\rho = P(p)S(s). \tag{5'}$$

The six dependent variables u_i, p, ρ, s characterise the flow. Two flows u_{i1}, p_1, ρ, s_1; u_{i2}, p_2, ρ_2, s_2, both of which satisfy (2′)-(5′) will be said to bo superposable or additive if it is possible to find p, P, s such that $u_{i1}+u_{i2}, p, \rho, s$ is also a solution of (2′)-(5′). *A* flow will be said to be self-superposable or self-additive if it is supersposable on itself. Thus if u_i are the velocity components for a self-superposable flow, then both u_i and $2u_i$ satisfy the equations (2′)—(5′), of course for different values of pressure, density and specific entropy.

For gases for which the equation of state is represented by (12) i.e. for those for which pressure can be eliminated from the basic equations, equation (13) together with equations (2)-(5) shows that *all steady motions for such gases are self-superposable.* In fact it shows that if u_i satisfy (13), then ku_i would also satisfy it whatever positive or negative constant value k takes, thus showing the essentially linear character of the fluid velocity for steady flow of such gases and the non-linear character of the pressure, density and entropy fields may be responsible for the non-linearity of the basic system of equations. The principle

of self-superposability established here for steady flows of gases with equation of state (12) is also valid for the case of the more general equation of state (5). This we proceed to show in the next section where we shall also find the necessary adjustments in pressure, density and entropy fields.

4. The substitution Principle for the Steady Motion of a Gas with Product Equation of State. If $Ku_j, K'\rho,\ p, s'$ satisfy the basic equations (2)-(5), we get from (7),

$$\frac{K^2 u_i u_j \left[K \dfrac{\partial u_i}{\partial x_j} + u_i \dfrac{\partial K}{\partial x_j} \right]}{K \dfrac{\partial u_j}{\partial x_j} + u_j \dfrac{\partial K}{\partial x_j}} = \frac{1}{K'\rho \dfrac{P'(p)}{P(p)}} \tag{15}$$

We now choose K as such a function that

$$u_j \frac{\partial K}{\partial x_j} = V \,.\, \text{grad}\, K = 0, \tag{16}$$

where $\mathbf{V}$ is the velocity vector i.e. K is a function which is constant along the streamlines of the original flow.

If both (7) and (15) are to hold

$$K' = \frac{1}{K^2} \tag{17}$$

Substituting $Ku_j, 1/K^2\,\rho,\ p, s'$ in (2)-(5) and using (16), we find

$$u_j \frac{\partial s'}{\partial x_j} = 0 \tag{18}$$

and

$$S(s') = \frac{1}{K^2}\,(Ss) \tag{19}$$

Thus we have proved *that if* $u_i,\ p, \rho, s$ *satisfy the basic equations* (2)-(5) *then* Ku_i, $p,\ 1/K^2\,\rho,\ s'$ *will also satisfy the basic equations where* K *is any function which is constant along the streamlines of the original (and therefore also of the new) flow and* s' *is determined from* (19) *In the new flow field also the entropy is constant along the streamlines. In the new flow, the pressure field is the same as that of the original flow and the Mach number distribution is the same*, for the local velocities of sound c, c' and the Mach numbers M, M' of the two flows are given by

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{1}{\rho}\ \frac{P(p)}{P'(p)}, \qquad c'^2 = \frac{1}{\rho/K^2}\,\frac{P(p)}{P'(p)} = K^2 c^2 \tag{20}$$

$$M = \frac{\sqrt{u_1^2+u_2^2+u_3^2}}{c}, \quad M' = K\frac{\sqrt{u_1^2+u_2^2+u_3^2}}{c'} = M. \tag{21}$$

Thus we have here an alternative proof of the substitution principle first established by Munk and Prim [29] for the steady flow of a non-viscous, thermally non-conducting gas with constant specific heats and later extended by Prim (1944) to the wider class of fluids having the product equation of state, viz;

"If in any flow field, the densities are multiplied by m and the velocity by $\frac{1}{m^{1/2}}$ where m is a positive function which is constant along each stream line, the new flow field will be a possible flow of the gas having the same pressure distribution and Mach number distribution as the original flow".

The equations (18) and (19) are important for they show that *if the original flow is isentropic, the new flow need not be isentropic unless K is a constant and if the original flow is non-isentropic it will be possible to get one flow from it which will be isentropic.* The last statement is made possible, since s and K are both constants along the streamlines so that K can be chosen as a suitable function of s, and choosing K to be proportional to $\sqrt{S(s)}$, we get $S(s') =$ constant, so that the new flow in this case is isentropic.

In particular by taking K to be a constant we get the result that *all steady flows of a non-viscous, non-heat-conducting gas with a product equation of state are self-superposable, superposed flow has the same pressure field as the original flow and the density here is $\frac{1}{K^2}$ times the density in the original flow where K is the ratio of the velocity in the superposed flow to that in the original flow. Moreover the two flows are either both isentropic or both non-isentropic.*

5. The Modified Substitution Principle for a Perfect Gas. For a gas for which $P(p) = p^{1/a}$, (7) gives

$$\frac{u_i u_j \dfrac{\partial u_i}{\partial x_j}}{\dfrac{\partial u_l}{\partial x_l}} = \frac{ap}{\rho} . \tag{22}$$

If both u_i, ρ, p and $Ku_i, K_1 \rho, K_2 p$ satisfy the basic equations and if K satisfies (16), then

$$K^2 = \frac{K_2}{K_1}, \; u_j \frac{\partial K_1}{\partial x_j} = 0, \; \frac{\partial K_2}{\partial x_i} = 0, \tag{23}$$

so that K_2 is constant. Thus for gases for which the equation of state is the product equation of state (5), but for which $P(p)$ is some power of p, we have a modified substitution principle viz that *if for such a gas u_i, p, ρ constitute a steady-state solution of the basic equations, then Ku_i, $\frac{K_2}{K^2}\rho$, $K_2 p$ also constitute a possible steady state solution where K_2 is a constant and K is a function which is constant along the streamlines of the two flows.* The pressure field is now changed, though only in a constant ratio, but the Mach number distribution is still unchanged. The relation (19) is modified to

$$S(s') = \frac{K_2^{\,1-\frac{1}{a}}}{K^2} S(s), \tag{24}$$

so that the result that either both flows are isentropic or non-isentropic remains unchanged.

By putting $K_2 = 1$ we derive the earlier substitution principle, as a particular case, though that principle was more general in another sense viz. that it held for a more general equation of state. The present principle, of course can be applied to a perfect gas for which $P(p) = p^{\frac{1}{\gamma}}$.

The advantage of the present pinciple is that it gives us a doubly infinite system of new solutions instead of the earlier principle which gave us only a single infinity of new solutions.

Thus we can find new solutions in which the distribution of any one of the variables is unchanged or modified in a given ratio. In particular, we have that if u_i, p, ρ, s constitute the steady state solution, then if K is a constant, and γ is the ratio of specific heats, the following are also solutions,

$$(i) \qquad u_i,\ K\rho,\ Kp,\ s';\ S(s') = K^{1-1/\gamma} S(s). \tag{25}$$

In this case the velocity field is unchanged

$$(ii) \qquad Ku_i,\ \rho,\ K^2 p,\ s'';\ S(s'') = K^{-2/\gamma} S(s). \tag{26}$$

In the case the density field is unchanged.

$$(iii) \qquad Ku_i,\ \frac{1}{K^2}\rho,\ p,\ s''';\ S(s''') = \frac{1}{K^2} S(s). \tag{27}$$

In this case the pressure field is unchanged.

$$(iv) \qquad Ku_i,\ K^{\frac{2}{\gamma-1}}\rho,\ K^{\frac{2\gamma}{\gamma-1}} p,\ s. \tag{28}$$

In this case the entropy is unchanged.

Alternatively we can apply the concept of self-superposability which we have so far applied only to the velocity field-to the other fields viz. the pressure, density and entropy fields also and say that for the steady frictionless flow of a perfect gas each of the fields-the pressure field, the density field and entropy field is self-superposable for if u_i, p, ρ, s is solution, then the following are also solutions :

$$(v) \qquad Kp,\ K\rho/m^2,\ mu_i, s^{iv},\ S(s^{iv}) = K^{1-\frac{1}{\gamma}}/m^2.\ S(s) \tag{29}$$

$$(vi) \qquad K\rho,\ K^2 mp,\ mu_i,\ s^{v};\ S(s^{v}) = K^{1-\frac{1}{\gamma}} m^{-\frac{2}{\gamma}} S(s) \tag{30}$$

$$(vii) \qquad Ks,\ m^{-\frac{\gamma-1}{2\gamma}} e^{-\frac{s-s_0}{c_p}\frac{1-k}{2}} u_i,\ m^{2-\frac{2}{\gamma}} e^{-\frac{s-s_0}{c_p}(m-1)} \rho,\ mp \tag{31}$$

Thus we may say that in the steady case, taken separately, each of the fields, pressure field, density field, entropy field is linear, but taken as a whole, the system is non-linear.

Another class of gases for which the pressure field can be varied are those for which

$$P(p) = e^{p^n} \ , \tag{32}$$

for in this case (7) becomes

$$\frac{u_i u_j u_{i,j}}{u_{l,\,l}} = \frac{1}{n\rho\, p^{n-1}} \tag{33}$$

If both u_i, p, ρ and $Ku_i, K_1' \rho, K_2' p$ satisfy (2), (3) and (33), where K satisfies (16), we get

$$K^2 = \frac{1}{K_1'(K_2')^{n-1}} \ , \ K_2' = \text{constant} \tag{34}$$

so that for such gases if u_i, p, ρ constitute a steady solution, then

$$Ku_i, \ K^{-2} K_2'^{1-n} \rho, \ K_2' p \tag{35}$$

also satisfy the equations of steady flows where K_2' is a constant and K is constant along the stream lines of the two flows.

6. **Motions Superposable on Steady Two-Dimentional Flows.** For steady isentropic flow of a perfect gas, the equations for determining the velocity components are :

$$-(\gamma-1)\, u_l\, u_{k,l} = \left(\frac{u_i u_j u_{i,j}}{u_{m,m}} \right)_{,k} \tag{36}$$

where a comma followed by a suffix denotes differentiation with respect to the variable corresponding to that suffix.

If $[u_1(x_1, x_2), u_2(x_1, x_2), 0]$ satisfies these equations we get

$$-(\gamma-1)[u_1 u_{1,\,1} + u_2\, u_{1,\,2}] = \left[\frac{u_1^2 u_{1,\,1} + u_2^2 u_{2,\,2} + u_1 u_2 (u_{1,\,2} + u_{2,\,1})}{u_{1,\,1} + u_{2,\,2}} \right]_{,1} \tag{37}$$

$$-(\gamma-1)[u_1 u_{2,\,1} + u_2 u_{2,2}] = \left[\frac{u_1^2 u_{1,1} + u_2^2 u_{2,2} + u_1 u_2 (u_{1,2} + u_{2,1})}{u_{1,1} + u_{2,2}} \right]_{,2} \tag{38}$$

If $[0, 0, u_3(x_1, x_2, x_3)]$ satisfies (36), then we get

$$u_{3,\,3} = 0 \tag{39}$$

so that u_3 has to be a function of x_1, x_2 alone.

If $[0, 0, u_3(x_1, x_3)]$ is superposable on $[u_1(x_1, x_2), u_2(x_1, x_2), 0]$ then $[u_1(x_1, x_2), u_2(x_1, x_2), u_3(x_1, x_2)]$ must satisfy (36). This gives

$$-(\gamma-1)[u_1, u_{1,1}+u_2u_{1,2}] = \left[\frac{u_1^2u_{1,1}+u_2^2u_{2,2}+u_1u_2(u_{1,2}+u_{2,1})+u_3(u_1u_{3,1}+u_2u_{3,2})}{u_{1,1}+u_{2,2}}\right]_{,1} \tag{40}$$

$$-(\gamma-1)[u_1\, u_{2,1}+u_2u_{2,2}] = \left[\frac{u_1^2u_{1,1}+u_2^2u_{2,2}+u_1u_2(u_{1,2}+u_{2,1})+u_3(u_1, u_{3,1}+u_2u_{3,2})}{u_{2,1}+u_{2,2}}\right]_{,2} \tag{41}$$

$$-(\gamma-1)[u_1\, u_{3,1}+u_2u_{3,2}]= 0 \tag{42}$$

If (42) is satisfied, then (40), (41) reduce to (37) and (38).

Thus the condition that the flow $[0, 0, u_3(x_1, x_2)]$ may be superposable on the two-dimensional flow $[u_1(x_1, x_2), u_2(x_1, x_2), 0]$ is

$$u_1u_{3,1}+u_2u_{3,2} = 0 \tag{43}$$

i.e. u_3 is constant along the stream lines of the two-dimensional flow so that u_3 is a function of ψ, where ψ is the stream function of the two-dimensional flow. Thus if

$$\vec{q_2} = u_1\vec{i_1}+u_2\,\vec{i_2} \tag{44}$$

gives a possible two-dimensional flow, then

$$\vec{q_3} = u_1\,\vec{i_1}+u_2\vec{i_2}+\phi(\psi)\vec{i_3} \tag{45}$$

is also a possible three dimensional flow of the same gas.

This, of course, includes the case when $\phi(\psi)$ is a constant, a case which Portisky (1945) obtained from Newtonian relativity considerations.

On making use of (43) and the fact that u_3 is not a function of x_3, we find from equations (2) and (3) that pressure and density fields are the same for both the flows (44) and (45). Also from (5), the new flow is isentropic.

We could have also expected this result, on intuitive grounds from Newtonian relativity considerations for if we have a steady two dimensional flow and if the fluid moves in a direction perpendicular to the plane of motion in such a way that the velocity is the same for points on the same stream line, though it may vary from streamline to streamline, the flow pattern in any plane parallel to the original plane of motion will remain unchanged. This consideration also shows that the principle will also be applicable in the case of incompressible flows and for both incompressible and compresible flows it gives a means for generating three-dimensional flows from two-dimensional flows.

The vorticity components of the three-dimensional motion are

$$u_{3,2}-u_{2,3}; \quad u_{1,3}-u_{3,1}, u_{2,1}-u_{1,2}$$

and these will vanish if

$$u_{3,2}=0 \; ; \quad u_{3,1}=0 \; ; \quad u_{2,1}-u_{1,2}=0 \tag{46}$$

i.e. if u_3 is constant and the original flow is irrotational. Thus in general, the new velocity field will be rotational even if the original flow is irrotational.

For seeing whether the principle is applicable for non-isentropic flows, we obtain the conditions that both the flows I and II satisfy the basic equations (2)-(5) where those flows are characterised by :

$$\text{I}: \quad u_1(x_1, x_2), u_2(x_1, x_2), 0 \; ; \quad p(x_1. x_2) \; ; \; \rho((x_1, x_2); \quad s(x_1, x_2). \tag{47}$$

$$\text{II}: \quad u_1(x_1, x_2), \; u_2(x_1, x_2), \; u_3(x_1, x_2) \; ; \; p(x_1, x_2) \; ; \; \rho(x_1 x_2) \; ; s(x, x_2). \tag{48}$$

If flow II satisfies the basic equations, then

$$\rho(u_{1,1}+u_{2,2})+u_1\rho_{,1}+u_2\rho_{,2}=0 \tag{49}$$

$$\rho(u_1 u_{1,1}+u_2 u_{1,2})=-p_{,1} \; ; \quad \rho(u_1 u_{2,1}+u_2 u_{2,2})=-p_{,2}; \quad \rho(u_1 u_{3,1}+u_2 u_{3,2})=0 \tag{50}$$

$$u_1 \; s_{,1}+u_2 s_{,2}=0 \tag{51}$$

If flow I is known to satisfy the basic equations, then flow II will also satisfy these equations if the additional condition

$$u_1 u_{3,1}+u_2 u_{3,2}=0 \tag{52}$$

is satisfied. (52) is, of course, the same as (43).

Thus we find that for both isentropic and non-isentropic two dimensional flows, a third velocity component $u_3(x_1, x_2)=\phi(\psi)$ normal to the original plane of flow can be superposed on the two dimensional flow field and the superposed flow will give a possible flow with the same pressure, density and entropy as in the original flow.

It is easily seen that the above principle will also hold for diabatic flow with Q_t the rate of heat addition, as a function of x_1, x_2.

7. Construction of Three-Demensional Exact Analytical Solutions from Two Dimensional Solutions and Properties of these Solutions. We now combine the principles of superposition and substitution to construct three-diemensional solutions from two-dimensional solutions, Thus let

$$\vec{q}_1 = u_1(x_1, x_2)\vec{i}_1 + u_2(x_1, x_2)\,\vec{i}_2 \tag{53}$$

be a two-dimensional motion with stream function ψ with pressure p and density ρ, then by the substitution principle, a more general two-dimensional flow is

$$\vec{q}_2 = f(\psi)u_1\,\vec{i}_1 + f(\psi)u_2\,\vec{i}_2\ , \tag{54}$$

and for this flow the pressure will be Kp, the density will be $K\rho/\{f(\psi)\}^2$ and the entropy s' will be determined from

$$S(s') = \frac{K^{1-1/\gamma}}{[f(\psi)]^2}\,S(s) \tag{55}$$

where K is any constant.

Now by using the superposition principle, the three dimensional flow,

$$\vec{q}_3 = f(\psi)u_1\vec{i}_1 + f(\psi)u_2\,\vec{i}_2 + \phi(\psi)\,\vec{i}_3 \tag{56}$$

will also be a possible flow with same pressure, density and entropy as for flow (54).

Moreover from Croccos' vortex theorem

$$\vec{q}\times \text{Curl}\,\vec{q} = \text{grad}\,(h+\tfrac{1}{2}\,q^2) - T\,\text{grad}\,s$$

or

$$\vec{q}\times\vec{\omega} = \text{grad}\,h_0 - T\,\text{grad}\,s,$$

where $\vec{\omega}$ is the vorticity vector, h is the enthalpy, h_0 is the stagnation enthalpy, and T is the absolute temperature.

For flow (53)

$$\vec{q}_1\times\vec{\omega}_1 = \text{grad}\left(\frac{\gamma}{\gamma-1}\,\frac{p}{\rho} + \frac{1}{2}q^2\right) - \frac{p}{\rho R}\,\text{grad}\ s \tag{58}$$

and for flow (56)

$$\vec{q}_3 \times \vec{w}_3 = 2\phi(\psi)\,\phi'(\psi)\ \text{grad}\,\psi + 2\left(\frac{\gamma}{\gamma-1}\,\frac{p}{\rho}+\frac{1}{2}q^2\right) f(\psi)\,f'(\psi)\ grad\,\psi$$

$$+(f(\psi))^2\ \text{grad}\left(\frac{\gamma}{\gamma-1}\,\frac{p}{\rho}+\frac{1}{2}q^2\right) \tag{59}$$

$$-\frac{p}{\rho}\,\frac{K^{1-\gamma}}{S'(s')}\left[S'(s)\ \text{grad}\ s - \frac{2}{f(\psi)}\,S(s)f'(\psi)\ \text{grad}\,\psi\right].$$

We now wish to investigate the conditions under which the new flow will be (i) irrotational (ii) Beltrami (iii) isoenergic (iv) isoentropic.

(a) The new flow will be irrotational if

$$\vec{\omega}_3 = \text{curl}\ \vec{q}_3 = \phi'(\psi)\psi_{,2}\vec{i}_1 - \phi'(\psi)\psi_{,1}\,\vec{i}_2$$

$$+[f'(\psi)(\psi_{,1}\,u_2-\psi_{,2}\,u_1)+f(\psi)(u_{2,1}-u_{1,2})]\,\vec{i}_3 = 0 \tag{60}$$

If the original flow is irrotational, the new flow will be irrotational only if both $f(\psi)$ and $\phi(\psi)$ are constants. If the originl flow is rotational, then the new flow will be irrotational if $\phi(\psi)$ is constant and $f(\psi)$ is determined from

$$f'(\psi)\ [\psi_{,1}\,u_2-\psi_{,2}u_1]+f(\psi)\,[u_{2,\,1}-u_{1,2}]=0 \tag{61}$$

For this to be possible it is necessary that

$$[\psi_{,1}u_2-\psi_{,2}u_1]/(u_{2,\,1}-u_{1,\,2})$$

should be a function of ψ alone. *Thus if the original flow is rotational, it is not in general possible to find a flow from it, which will be irrotational. In one particular case, however, it will always be possible viz. in the case when the velocity magnitude is constant along the stream lines* for in this case density, pressure, temperature would also be constant along the streamlines and in the two-dimensional case the vorticity is also constant along streamlines. [Kapur A].

(b) *The new flow will be isentropic if (i) either the original flow is isentropic and $f(\psi) = constant$* or

(ii) *if $[f(\psi)]^2$ is chosen proportional to $S(s)$ which* itself is a function of ψ since s is constant along the stream lines and is thus a function of ψ.

(c) *The new flow will be isoenergic if*

either (i) the original flow is isoenergic and $\phi(\psi) =$ constant or (ii) $[f(\psi)]^2h_0+[\phi(\psi)]^2 =$ *constant* $= A$ *(say),*

where h_0 is the original stagnation enthalpy. Since h_0 is a function of ψ, it is possible to choose $f(\psi)$ and $\phi(\psi)$ to satisfy (62), but the constant stagnation enthalpy A has to be such that $A-h_0\,[f(\psi)]^2$ is positive throughout the flow region. Thus given A, the function $f(\psi)$ cannot be completely arbitrary or given $f(\psi)$, A cannot be arbitrary, but A must be greater than the maximum value of $h_0[f(\psi)]^2$ in the flow region.

(*d*) The new flow will be both isentropic and isoenergic of $[f(\psi)]^2$ is proportional to $S(s)$ and (62) is also satisfied. Thus both $f(\psi)$ and $\phi(\psi)$ will be determined. Thus, *given a two-dimensional flow which may be neither isentropic nor isoenergic it is possible from it to generate a three-dimensional flow which will be both isentropic as well as isoenergic. From Crocco's vortex theorem it then follows that we can always generate a three-dimensional Beltrami flow from a given two-dimensional flow.*

(*e*) For a flow to be Beltrami, it is sufficient but not necessary that it should be both isentropic and isoenergic. Thus if the new flow is Beltrami, the necessary condition is

$$\text{grad}\ [\{F(\psi)\}^2h_0+\{\phi(\psi)\}^2]-T(f(\psi))^2\ \text{grad}\ s' = 0, \tag{63}$$

where

$$S(s') = \frac{K^{1-\frac{1}{\gamma}}}{[f(\psi)]^2}\ S(s) \tag{64}$$

Since T is, in general, not a function of ψ, it follows that *in all cases where T is not a function of ψ, the new flow will be Beltrami only if it is both isentropic and isoenergic. If, however, the velocity magnitude is constant along streamlines in the original flow i.e. if in the original flow the streamlines are concentric circles or parallel straight lines,* [Kapur A] *then T will also be constant along the streamlines and this will be a function of ψ. In this case* (63) *gives the only restriction on $f(\psi)$ and $\phi(\psi)$.*

(*f*) The density in the new flow is $\rho/[f(\psi)]^2$ and $f(\psi)$ can be chosen to make it constant provided ρ is constant along the stream lines of the original flow. Similarly the square of the velocity in the new flow is $[f(\psi)]^2q^2+[\phi(\psi)]^2$ where q is the velocity in the original flow. This can also be made constant in the new flow provided q is constant along the stream lines of the original flow. Similarly since the temperature in the new flow is $T[f(\psi)]^2$, it is possible to find a flow in which the temperature is throughout uniform provided in the original flow the temperature is constant along the stream lines. *Thus we find that if in a two or three-dimensional flow the velocity magnitude is constant along streamlines, then it is possible to generate from them three dimensional flows for which either (i) the velocity magnitude is constant throughout the region of flow or (ii) the density is constant throughout this region of flow or (iii) the temperature is constant throughout the region. The second case is of special interest as it provides a means of connecting compressible and incompressible flows.*

Thus we find that flow fields in which velocity magnitude is constant along streamlines are of special significance. Prim [38] proved that for two-dimensional flows of this type, streamlines must be either concentric circles or straight lines and axially-symmetric flows of this type must be purely axial. Alternative proofs of these facts together with a discussion of the three-dimensional case have been given by Kapur [A].

8. Some Particular Examples. (*a*) *Generalised vortex flow* :

For a two-dimensional vortex of straight K,

$$\vec{q} = 0 \,.\, \vec{i_r} + \frac{K}{2\pi}\,\vec{i_\theta} + 0 \,.\, \vec{i_z} \,. \tag{65}$$

Since the streamlines are circles, for the generalised flow

$$\vec{q} = \frac{f(r)}{r}\vec{i_\theta} + \phi(r)\vec{i_z}. \tag{66}$$

The streamlines are the intersections of the surfaces,

$$r = \text{const}, \quad f(r)r^{-2}\,z - \phi(r)\theta = \text{const}. \tag{67}$$

i.e. they are right circular helices.

If p, ρ, s denote the pressure, density, entropy of the original field, then for the new flow the pressure is proportional to p, the density is proportional to $\rho/[f(r)]^2$ and the entropy $s'-s_0'$ is proportional to $-\log\,[f(r)]^2$. Since for the original flow the density vanishes inside a certain circle, for the new flow also it will be vanish inside a circular cylinder except in one case i.e. when $f(r)$ is chosen as $\surd\rho$, so that the density is throughout uniform. The pressure will increase with r.

(*b*) *Generalised radial flow* :

For two-dimensional radial flow.

$$\vec{q} = \frac{\sigma}{2\pi r(\rho/\rho_0)}\;\vec{i_r} + 0.\,\vec{i_\theta} + 0 \,.\, \vec{i_z}, \tag{68}$$

where σ is a measure of the strength of the source and ρ/ρ_0 is given by

$$r^2 = \frac{\sigma^2}{4\pi^2 q^2_{max}}\left(\frac{\rho}{\rho_0}\right)^2 \frac{1}{[1-(\rho/\rho_0)^{\gamma-1}]} \tag{69}$$

The generalised flow is

$$\vec{q} = \frac{f(r)\;\sigma}{2\pi r\,(\rho/\rho_0)}\,\vec{i_r} + 0 \,.\, \vec{i_\theta} + \phi(r)\vec{i_z} \,. \tag{70}$$

We can choose $f(r)$, $\phi(r)$ so that in the new field either 'the velocity magnitude or the density or the temperature are constant throughout.

(c) *Generalised Prandtl-Meyer Flow.*

Here for two-dimensional flow

$$\vec{q} = q_{max} \sin \lambda\theta \, \vec{i_r} + \lambda q_{max} \cos \lambda\theta \, \vec{i_\theta} + 0. \vec{i_z} \tag{71}$$

$$\frac{\rho}{\rho_s} = \cos^{2\beta}\lambda\theta, \frac{p}{p_s} = \cos^{2\beta+2}\lambda\theta, \frac{a}{a_s} = \cos \lambda\theta, \frac{T}{T_s} = \cos^2 \lambda\theta \tag{72}$$

$$\psi = -\lambda q_{max} r \cos^{2\beta+1}\lambda\theta \tag{73}$$

where

$$\lambda^2 = \frac{\gamma-1}{\gamma+1}, \quad \beta = \frac{1}{\gamma-1}, 2\beta+1 = \frac{1}{\lambda^2} \tag{74}$$

The generalised flow is

$$\vec{q} = f(\psi) q_{max} \sin \lambda\theta \, \vec{i_r} + f(\psi)\lambda q_{max} \cos \lambda\theta \, \vec{i_\theta} + \phi(\psi) \vec{i_z} \tag{75}$$

The stream lines are given by

$$\psi = -q_{max} r \cos^{2\beta+1}\lambda\theta = \text{const} \tag{76}$$

$$z = \text{const} \int \sec^{\frac{2\gamma}{\gamma-1}} \lambda\theta \, d\theta \tag{77}$$

The properties of the above three flows and other three-dimensional flows that can be generated from known two-dimensional flows by using the methods of the present paper will be discussed separately in a paper which will also deal with the axially-symmetric case.

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LAMINAR FLOW OF NON-NEWTONIAN FLUID IN CHANNELS WITH POROUS WALLS

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In this paper laminar flow of a certain type of non-Newtonian fluid between two porous plane parallel boundaries has been discussed assuming constant suction at both the walls. The type of fluid taken satisfies the stress-strain relation postulated by Rivlin (1948).

Introduction. Laminar flow of a Newtonian viscous fluid near the entrance end of channels of plane parallel porous walls has been considered by Berman and Yuan for small values of the cross flow Reynold's number R (Berman, 1953) and also for large values of R (Yuan, 1956). In both the cases a suitable form of the stream function has been chosen and then solution has been obtained as a series in R (if R is small) or as a series in R^{-1} (if R is large). In our case we have taken the same form of the stream function and have shown that for such a choice the flow pattern does not depend on the cross-viscous coefficient, though the pressure drops in the axial direction as also in the transverse direction depend on the cross-viscous coefficient. This means that the normal stresses will also depend on the same. In a previous communication (Dutta, 1960) we studied the same problem for a fully developed laminar flow.

Equations. The stress, rate of strain relation for a certain non-Newtonian fluid as postulated by Rivlin (1948) is

$$\begin{aligned} \tau_{ij} &= -p\delta_{ij}+2\mu e_{ij}+4\mu_c e_{ik}e_{kj}, \\ e_{ij} &= \tfrac{1}{2}(v_{i,j}+v_{j,i}), \\ \delta_{ij} &= 1,\ i=j\ ;\ =0,\ i\neq j, \end{aligned} \tag{1}$$

where μ is the viscous parameter and μ_c the cross-viscous parameter. This equation can also be obtained from the Oldroyd's formulation (Oldroyd, 1958) for an elastico-viscous non-Newtonian fluid by the choice of the suitable values of the constants.

We consider a channel of rectangular section, the distance between the porous planes being taken to be very small in comparison with the longitudinal dimensions of the walls. This condition enables us to treat the problem as a two-dimensional one. Both channel walls are taken to be of equal permeability, so that the velocity of the fluid leaving the walls may be taken as independent of the distance from the entrance end. We choose Cartesian co-ordinates with origin midway between the walls y-axis perp. to the walls and x-axis parallel to flow. Taking $\lambda = y/h$, $2h$ being the distance between the walls. We have

$$u\frac{\partial u}{\partial x}+\frac{v}{h}\frac{\partial u}{\partial \lambda}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{1}{h^2}\frac{\partial^2 u}{\partial \lambda^2}\right]+$$

$$+\nu_c\frac{\partial}{\partial x}\left[4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{1}{h}\frac{\partial u}{\partial \lambda}+\frac{\partial v}{\partial x}\right)^2\right],$$

$$u\frac{\partial v}{\partial x}+\frac{v}{h}\frac{\partial v}{\partial \lambda}=-\frac{1}{h\rho}\frac{\partial p}{\partial \lambda}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{1}{h^2}\frac{\partial^2 v}{\partial \lambda^2}\right]+$$

$$+\frac{\nu_c}{h}\frac{\partial}{\partial \lambda}\left[\frac{4}{h^2}\left(\frac{\partial v}{\partial \lambda}\right)^2+\left(\frac{1}{h}\frac{\partial u}{\partial \lambda}+\frac{\partial v}{\partial x}\right)^2\right], \tag{2}$$

where $\mu/\rho=\nu$, $\mu_c/\rho=\nu_c$ and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{1}{h}\frac{\partial v}{\partial \lambda}=0. \tag{3}$$

But by virtue of our foregoing assumptions we may take

$$\frac{\partial v}{\partial x}=0. \tag{4}$$

The boundary conditions are

$$u(x,\pm 1)=0, \tag{5}$$

$$\left.\frac{\partial u}{\partial \lambda}\right|_{\lambda=0}=0 \tag{6}$$

$$v(x,0)=0, \tag{7}$$

$$v(x,\pm 1)=v_w=\text{constant}. \tag{8}$$

Since the flow is symmetrical about a plane midway between the porous walls, we consider the flow over half the channel, i.e. from the middle to one wall. We take the stream function as

$$\psi(x,\lambda)=[h\bar{u}(0)-v_w x]f(\lambda), \tag{9}$$

where $\bar{u}(0)$ is the average velocity (axial) at $x = 0$. So

$$u(x, \lambda) = [\bar{u}(0) - v_w x/h] f'(\lambda). \tag{10}$$

$$v(\lambda) = v_w f(\lambda). \tag{11}$$

Substituting (10) and (11) in (2) and using (4) we obtain

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\left[\bar{u}(0) - \frac{v_w x}{h}\right]\left[\frac{v_w}{h}\left(f'^2 - ff'' - \frac{2\nu_c}{h^2} f''^2\right) + \frac{\nu}{h^2} f'''\right], \tag{12}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial \lambda} = v_w^2\left[ff' - \frac{\nu}{hv_w} f''\right] - \frac{\nu_c}{h^2}\frac{\partial}{\partial \lambda}\left[4v_w^2 f'^2_w\{\bar{u}(0) - v_w x/h\}^2 f''^2\right] \tag{13}$$

and the boundary conditions (5)—(8) give

$$f(0) = f'(1) = f''(0) = 0, \quad f(1) = 1. \tag{14}$$

From (12) and (13) by a suitable reduction, we get

$$R[f'^2 - ff''] + f''' = C, \tag{15}$$

where $R = v_w h/\nu$ and C is a constant. This is the same equation as obtained by Berman and Yuan.

Case I. So if v_w is small i.e. R is small, we assume following Berman

$$f = f_0 + Rf_1 + R^2 f_2 + \dots \quad , \tag{16}$$

$$C = C_0 + RC_1 + R^2 C_2 + \dots \quad , \tag{17}$$

with the appropriate boundary conditions

$$f_n(0) = f_n'(1) = f_n''(0) = 0, \qquad n \geqslant 0,$$

$$f_0(1) = 1\,; \qquad f_n(1) = 0, \qquad n \geqslant 1.$$

The first order perturbation solutions for $f(\lambda)$ and C are then (Berman, 1953)

$$f^{(1)}(\lambda) = \frac{1}{2}\lambda(3 - \lambda^2) + \frac{R}{280}\,(3\lambda^3 - 2\lambda - \lambda^7),$$

$$C^{(1)} = -3 + \frac{81}{35} R. \tag{18}$$

The expression for the axial pressure drop can be obtained from the equation (13) in the form

$$\frac{p(0,\lambda)-p(x,\lambda)}{\frac{1}{2}\rho\bar{u}^2(0)}=-\frac{x}{hN_{Re}}\left[8-\frac{16R}{N_{Re}}\left(\frac{x}{h}\right)\right]$$

$$\times\left[-3+\frac{81}{35}R-18K\lambda^2+18KR\lambda^2\left(\frac{3}{70}-\frac{\lambda^4}{10}\right)\right], \tag{19}$$

where

$$K=\frac{\nu_c v_w}{h\nu}=\frac{\nu_c R}{h^2}.$$

Thus if ν_c is finite and $\approx 0(\nu)$, we may neglect the product KR in (19) and write

$$\frac{p(0,\lambda)-p(x,\lambda)}{\frac{1}{2}\rho\bar{u}^2(0)}=-\frac{1}{N_{Re}}\left(\frac{x}{h}\right)\left[8-\frac{16R}{N_{Re}}\left(\frac{x}{h}\right)\right]\left[-3+\frac{81}{35}R-18K\lambda^2\right], \tag{20}$$

where N_{Re} is the entrance Reynold's number $=4h\bar{u}(0)/\nu$. This expression contains K and so depends on the cross-viscous coefficient μ_c. Thus, although the flow pattern does not depend on μ_c, the pressure drop is a function of the same. We can also see that the stress components τ_{xy}, τ_{xx}, τ_{yy} at the walls are given by,

$$[\tau_{xy}/\tfrac{1}{2}\bar{u}(0)^2\rho]_{\lambda=\pm1}=\frac{8}{N_{Re}}\left[1-\frac{4R}{N_{Re}}\left(\frac{x}{h}\right)\right]\left[-3\lambda+\frac{R}{140}(3\lambda-21\lambda^5)\right]_{\lambda=\pm1},$$

$$[\tau_{xx}/\tfrac{1}{2}\rho\bar{u}(0)^2]_{\lambda=\pm1}=-\frac{[p]_{\lambda=\pm1}}{\frac{1}{2}\rho\bar{u}(0)^2}-\frac{64R}{N_{Re}^2}\left\{\frac{3}{2}(1-\lambda^2)+\frac{R}{280}(9\lambda^2-7\lambda^6-2\right\}_{\lambda=\pm1}+$$

$$+8K\left[\frac{16R}{N_{Re}^2}\left\{\frac{3}{2}(1-\lambda^2)+\frac{R}{280}(9\lambda^2-7\lambda^6-2)\right\}^2+\right.$$

$$\left.+\frac{1}{R}\left\{1-\frac{4R}{N_{Re}}\left(\frac{x}{h}\right)\right\}^2\left\{-3\lambda+\frac{R}{140}(3\lambda-21\lambda^5)\right\}\right]_{\lambda=\pm1}$$

$$[\tau_{yy}/\tfrac{1}{2}\rho\bar{u}(0)^2]_{\lambda=\pm1}=-\frac{[p]_{\lambda=\pm1}}{\frac{1}{2}\rho\bar{u}(0)^2}+\frac{64R}{N_{Re}^2}\left\{\frac{3}{2}(1-\lambda^2)+\frac{R}{280}(9\lambda^2-7\lambda^6-2)\right\}_{\lambda=\pm1}+$$

$$+8K\left[\frac{16R}{N_{Re}^2}\left\{\frac{3}{2}(1-\lambda^2)\frac{R}{280}(9\lambda^2-7\lambda^6-2)\right\}^2\right.$$

$$+\frac{1}{R}\left\{1-\frac{4R}{N_{Re}}\left(\frac{x}{h}\right)\right\}^2.$$

$$\left.\times\left\{-3\lambda+\frac{R}{140}(3\lambda-21\lambda^5)\right\}^2\right]_{\lambda=\pm1},$$

where

$$\frac{-1}{\frac{1}{2}\rho\bar{u}(0)^2}\,[p(\lambda, x)-p(0,0)]=\frac{1}{R}\left[1-\frac{4R}{N_{Re}}\left(\frac{x}{h}\right)\right]^2\left[-3+\frac{81}{35}R\right]-\frac{16R}{N_{Re}^2}(3-2\lambda)-$$

$$-\frac{2K}{R}\left[1-\frac{4R}{N_{Re}}\left(\frac{x}{h}\right)\right]^2\left[9\lambda^2-18R\left\{\frac{3\lambda}{140}-\frac{\lambda^5}{20}\right\}\right]+C,$$

where C is an additive constant. Here we have neglected powers of R higher than 1 and the products KR etc.

This shows that τ_{xy} is independent of K, and τ_{xx} and τ_{yy} depend on the same.

Case II. If R is large, then following Yuan we write the equation (15) as

$$f'^2-ff''+\frac{1}{R}f''' = \alpha, \tag{21}$$

and assume as a solution

$$f = f_0+\frac{1}{R}f_1+\frac{1}{R^2}f_2+\cdots, \tag{22}$$

$$\alpha = \alpha_0+\frac{1}{R}\alpha_1+\frac{1}{R^2}\alpha_2+\cdots \tag{23}$$

Substituting (22) and (23) in (21) we obtain as the first order perturbation solution of (21) as (Yuan, 1956)

$$f^{(1)}(\lambda) = \sin\frac{\pi}{2}\lambda+\frac{1}{R}\Bigg[\left\{1\cdot438+\frac{\pi^3}{16}\int_1^\lambda\frac{\xi d\xi}{\sin\frac{\pi\xi}{2}}\right\}\cos\frac{\pi\lambda}{2}+$$

$$+\left\{0\cdot662+\frac{\pi^2}{8}\log\left|\tan\frac{\pi\lambda}{4}\right|\right\}\left\{\frac{2}{\pi}\sin\frac{\pi\lambda}{2}-\lambda\cos\frac{\pi}{2}\lambda\right\}-\frac{1\cdot324}{\pi}\sin\frac{\pi\lambda}{2}\Bigg], \tag{24}$$

$$\alpha^{(1)} = -\frac{\pi^2}{4}+\frac{2\cdot049}{R}. \tag{25}$$

Thus the expression for the axial pressure drop is

$$\frac{p(0,\lambda)-p(x,\lambda)}{\frac{1}{2}\rho\bar{u}(0)^2}=\frac{8R}{N_{Re}}\left(\frac{x}{h}\right)\left[1-\frac{2R}{N_{Re}}\left(\frac{x}{h}\right)\right]\left[\frac{\pi^2}{4}-\frac{2\cdot049}{R}-\frac{K\pi^2}{8R}\sin^4\frac{\pi}{2}\lambda\right], \tag{26}$$

neglecting $1/R^2$ and terms of the same order.

This also depends on K i.e. on ν_c. We can obtain the expressions for the stress components as in the previous case and show that τ_{xx} and τ_{yy} are dependent on K.

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PARALLEL FLOW IN AN ANNULAR CHANNEL IN HYDROMAGNETICS-I

By

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Abstract. In the present paper, the steady, incompressible flow of conducting viscous fluid in an annular channel with fixed boundaries, in the presence of radial magnetic field has been discussed. Changes in the velocity profile are shown with the variations in the (i) strength of the magnetic field (proportional to β) (ii) distance between the two cylinders (proportional to θ).

A pair of the points of inflexion in the flow region may exist and the condition for their existence has been obtained, velocity profiles are asymmetrical and it has been shown that the interaction between viscous and induction drags causes the shift of the point of maximum velocity towards the outer boundary. Induction drag is dominant over the viscous drag in the proximity of the inner cylinder, while the case is reversed near the outer cylinder. This gives rise to a sharp velocity drop near the outer boundary,

Introduction. Recently Pai (1954) has discussed the flow of a conducting fluid in a tube in the presence of a radial magnetic field and pointed out the similarity in pressure distribution with the corresponding tubulent flow in hydrodynamics. Globe (1959) has discussed similar problem in a cylindrical annulus. Magnetic field has been taken to be radial but is different in the form from Pai's field. The Globe's problem does not appear to have been discussed in full. He has obtained the solution of the system and the limiting form of the solution for parallel plates. The behaviour of the solution is quite interesting which has been discussed in the present paper.

It has been shown that for a given value of θ, there exist, in general, a critical value of $\beta(=\beta_c)$ such that for all values of $\beta > \beta_0$ there occurs a pair of points of inflexion in the flow region. Similar result is true if β is fixed. The velocity profiles are asymmetric and the point of maximum velocity occurs nearer to the outer boundary. This fact is contrary to the results in the absence of magnetic field. A physical interpretation has been provided for it. There also occurs a boundary layer near outer cylinder for large values of β and θ.

The solution of the problem with moving boundaries and their limiting cases have been considered in a separate paper. [Kapur and Jain (1962) Part II of the present paper] The behaviour of the solutions has also been discussed.

Notation and the Solution. An infinitely long cylindrical channel of a and b as its inner and outer radii is considered. An external magnetic field $(\omega/r,\ 0, 0)$ is present. Here ω is a constant and characterises the magnitude of the magnetic field. Cylindrical coordinates (r, θ, z) of a point have been used throughout the present paper. The common axis of the cylinders has been taken as z-axis.

On the assumptions that (i) fluid is viscous and electrically conducting (2) σ is so large as to neglect displacement currents and convection currents (3) free charge density is zero (4) μ (permeability) and σ (electrical conductivity) are constant (5) Lorentz force is the only body force (6) the flow is steady and incompressible (7) axial symmetry is prevalent; (8) flow is parallel to the axis: $\bar{v} = (o, o, v_z)$, and the applied magnetic field ω/r fixes the normal component of the magnetic field at $r = a$ and at $r = b$ for all values of z, and this is the only field impressed, it has been shown that (Globe, 1959, 1960, Kapur and Jain, 1960)

$$\text{(i)} \quad v_z = v_z(r)$$

$$\text{(ii)} \quad \bar{H} = \left[\frac{\omega}{r}, o, H_z(r)\right]$$

on the assumption that $H_z = f(z)+\phi(r)$ or $H_z = f(z)\,\phi(r)$. The solution of the problem is (Globe, 1959)

$$V_z = \frac{Pa^2}{(\beta^2-4)\eta}\left\{\gamma^2 - \frac{\theta^2 \sinh(\beta \ln \gamma) - \sinh(\beta \ln \gamma/\theta)}{\sinh(\beta \ln \theta)}\right\} \tag{1}$$

$$H_z = -\frac{Pa^2}{(\beta^2-4)\eta}\frac{\omega}{\lambda}\left\{\frac{\gamma^2-\theta^2}{2} + \frac{\cosh(\beta \ln \gamma/\theta) - \theta^2\cosh(\beta \ln \gamma) - 1 + \theta^2 \cosh(\beta \ln \theta)}{\beta \sinh(\beta \ln \theta)}\right\} \tag{2}$$

where $\beta^2 \neq 4$, and

$$(V_z)_{\beta^2=4} = \frac{Pa^2}{4\eta(\theta^4-1)}\left\{\gamma^2 \ln \gamma - \theta^4\gamma^2 \ln \gamma/\theta - \frac{\theta^4}{\gamma^2}\ln \theta\right\} \tag{3}$$

$$(H_z)_{\beta^2=4} = \frac{Pa^2}{4(\theta^4-1)\eta}\frac{\omega}{\lambda}\left\{\frac{\theta^4\gamma^2}{2}\ln \gamma/\theta - \frac{\theta^4}{2\gamma^2}\ln \theta - \frac{\gamma^2 \ln \gamma}{2} + \theta^2 \ln \theta + (\theta^4-1)\left(\frac{\theta^2-\gamma^2}{4}\right)\right\}. \tag{4}$$

The equations (1) and (2) can alternatively be written as

$$V_z = \frac{Pa^2}{(\beta^2-4)\eta}\left\{\gamma^2 - \frac{(\theta^2-\theta^{-\beta})\gamma^\beta + (\theta^\beta - \theta^2)/\gamma^\beta}{\theta^\beta - \theta^{-\beta}}\right\} \tag{5}$$

$$H_z = -\frac{Pa^2}{(\beta^2-4)\eta}\frac{\omega}{\lambda}\left\{\frac{\gamma^2-\theta^2}{2} + \frac{\gamma^\beta(\theta^{-\beta}-\theta^2) + \gamma^{-\beta}(\theta^\beta - \theta^2) - 1 + \theta^2(\theta^\beta + \theta^{-\beta})}{\beta(\theta^\beta - \theta^{-\beta})}\right\} \tag{6}$$

where $-P$ is a constant pressure gradient in the radial direction,

$$\gamma = r/a, \quad \theta = b/a, \quad \eta = \rho_\nu, \quad \beta^2 = \mu^2\omega^2\sigma/\eta \text{ and } \lambda = 1/(4\pi\mu\sigma)$$

Existence of the Points of Inflexion. The existence of the points of inflexion in the velocity profile in the flow region is generally connected with the problem of the stability of the flow. Therefore it is desirable to study their existence before discussing the stability problem. The necessary and sufficient condition for the existence of a point of inflexion at $\gamma = \xi$ is

$$\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=\xi} = 0 \text{ and } \left(\frac{d^3v_z}{d\gamma^3}\right)_{\gamma=\xi} \neq 0, \text{ where } 1 < \xi < \theta. \tag{7}$$

The second derivative of the velocity is

$$\frac{d^2v_z}{d\gamma^2} = \frac{k}{\beta^2-4}\left[2-\frac{\beta(\beta-1)\gamma^{\beta-2}(\theta^2-\theta^{-\beta})+\beta(\beta+1)(\theta^\beta-\theta^2)/\gamma^{\beta+2}}{\theta^\beta-\theta^{-\beta}}\right] \tag{8}$$

where $k = Pa^2/\eta,\ \beta^2 \neq 4,$ so that

$$\left(\frac{d^2v_z}{d\gamma^2}\right)_{\beta=0} = -\frac{k}{4}\left[\frac{\theta^2-1}{\ln\theta}\cdot\frac{1}{\gamma^2}+2\right] \tag{9}$$

and

$$\left(\frac{d^2v_z}{d\gamma^2}\right)_{\beta=2} = \frac{k}{4(\theta^4-1)}\left[2\theta^4\ln\theta\left(1-\frac{3}{\gamma^4}\right)-(\theta^4-1)(3+2\ln\gamma)\right]. \tag{10}$$

First, we shall determine the zeros of $\frac{d^2v_z}{d\gamma^2}$. *As* $\theta > 1$, $\frac{d^2v_z}{d\gamma^2}$ from (9) is always negative and therefore velocity profile, in the absence of the magnetic field, does not have any point of inflexion. For the zeros of $\frac{d^2v_z}{d\gamma^2}$ given by expression (8), one has

$$2A\gamma^{\beta+2}-B\gamma^{2\beta}-C = 0 \tag{11}$$

where $A = \theta^\beta-\theta^{-\beta},\ B = \beta(\beta-1)(\theta^2-\theta^{-\beta}),$ and $C = \beta(\beta+1)(\theta^\beta-\theta^2).$

If $\beta > 2$, then

$$A > 0,\ B > 0,\ C > 0.$$

In addition, if β is a positive integer greater than 2, left hand side of the equation (11) will represent a polynomial

$$f(\gamma) \equiv -B\gamma^{2\beta}+2A\gamma^{\beta+2}-C$$

in descending powers of γ. Clearly it has two changes of sign and therefore can have at the most two positive real zeros. Even if $\beta = p+\frac{1}{n}$, where p and n are any positive integers and $p \geqslant 2$, $f(\gamma)$ can be made to represent a polynomial

$$f(\varkappa) \equiv -B\kappa^{2(1+pn)}+2A\kappa^{1+pn}-C$$

by the transformation $\gamma^{\frac{1}{n}} = \kappa$ and the above result is still true. In fact $f(\gamma)$ corresponds to the flow which has to behave regularly . Therefore by the continuity of argument, $f(\gamma)$ can have at the most two real positive zeros for all real values of $\beta > 2$.

If $\beta = 1/n$ where n is any positive integer, one has

$$A > 0, \quad B \leqslant 0, \; C < 0,$$

equality for B being true when $n = 1$. Using the same transformation, the polynomial in descending powers of κ becomes

$$-B\kappa^2+2A\kappa^{1+2n}-C,$$

which does not have any change of sign. Therefore for $\beta \leqslant 1$, there is no zero of $f(\gamma)$ for all values of θ.

Similarly, if $\beta = 1+1/n$, n being any positive integer other than 1, it can be shown that $f(\gamma)$ can be represented as a polynomial and shall have two changes of sign. Thus one sums up that for all values of $\beta > 1$, $f(\gamma)$ has atmost two positive real zeros for all values of θ and for $\beta \leqslant 1$, $f(\gamma)$ has no zero. Obviously the flow patterns do not have any point of inflextion for all θ and $\beta \leqslant 1$.

Next question to be decided is about the position of the zeros. We consider

$$\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=1} = \frac{k}{(\beta^2-4)(\theta^\beta-\theta^{-\beta})}\left[B_1(\theta^{-\beta}-\theta^\beta)+2\beta(\theta^2-\theta^\beta)\right] \quad \ldots \quad (12)$$

and

$$\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=\theta} = -\frac{k}{(\beta^2-4)(\theta^\beta-\theta^{-\beta})}\left[B_1(\theta^\beta-\theta^{-\beta})+2\beta(\theta^{-2}-\theta^{-\beta})\right] \quad \ldots \quad (13)$$

where $\qquad B_1 = \beta^2-\beta-2 \qquad$ and $\quad \beta^2 \neq 4.$

Also $\qquad B_1 \lesseqgtr 0 \quad$ when $\quad \beta \lesseqgtr 2.$

One observes that for all values of θ and $\beta^2 \neq 4$, $\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=1}$ and $\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=\theta}$ are both negative. Moreover when $\beta^2 = 4$

$$\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=1} = -\frac{k}{4(\theta^4-1} \left[4\theta^4 \ln\theta+3(\theta^4-1)\right] \quad \ldots \quad (14)$$

and

$$\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=\theta} = -\frac{k}{4(\theta^4-1)}\left[4\ln\theta+3(\theta^4-1)\right]. \quad (15)$$

Here again $\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=\theta}$ and $\left(\frac{d^2v_z}{d\gamma^2}\right)_{\gamma=\theta}$ are negative for all values of θ. Thus both the zeros of $f(\gamma)$ will be either within the annulus or outside it.

To decide finally that the two zeros of $\frac{d^2v_z}{d\gamma^2}$ are points of inflexion, one has to consider the third order derivative of v_z,

$$\left(\frac{d^3v_z}{d\gamma^3}\right)_{\beta^2\neq 4}=\frac{k}{(\beta^2-4)(\theta^\beta-\theta^{-\beta})}\left[\frac{\beta(\beta+1)(\beta+2)}{\gamma^{\beta+3}}(\theta^\beta-\theta^2)-\beta(\beta-1)(\beta-2)\gamma^{\beta-3}(\theta^2-\theta^{-\beta})\right] \quad (16)$$

and

$$\left(\frac{d^3v_z}{d\gamma^3}\right)_{\beta^2=4}=-\frac{k}{2\gamma^5}\left[\gamma^5-\frac{15\theta^4\ln\theta}{(\theta^4-1)}\right]. \quad (17)$$

The zeros of $\frac{d^3v_z}{d\gamma^3}$ are given by

$$\gamma^{2\beta}=\frac{(\beta+1)(\beta+2)}{(\beta-1)(\beta-2)}\,\frac{\theta^\beta-\theta^2}{\theta^2-\theta^{-\beta}} \quad (18)$$

when $\beta^2\neq 4$ and

$$\gamma^5=\frac{15\theta^4\ln\theta}{\theta^4-1}$$

when $\beta^2=4$. One observes that right hand sides of equations (18) [for all values of β except for $\beta=1$, for which there, obviously exist no zero of $\frac{d^3v_z}{d\gamma^3}$] and (19) are positive. Thus in the case of our interest [$\beta>1$], there exist at most one real positive zero of $\frac{d^3v_z}{d\gamma^3}$. This zero of $\frac{d^3v_z}{d\gamma^3}$ cannot coincide with the zero of $\frac{d^2v_z}{d\gamma^2}$. It is because $\frac{d^2v_z}{d\gamma^2}$ has two zeros and since function is continuous there must occur an extremum of $\frac{d^2v_z}{d\gamma^2}$ between its two zeros where necessarily $\frac{d^3v_z}{d\gamma^3}$ has to vanish. To conclude, we have proved that :

(i) for $\beta\leqslant 1$, there is no point of inflexion,

and (ii) for $\beta>1$, either there is no point of inflexion or there are two points of inflexion within the flow region.

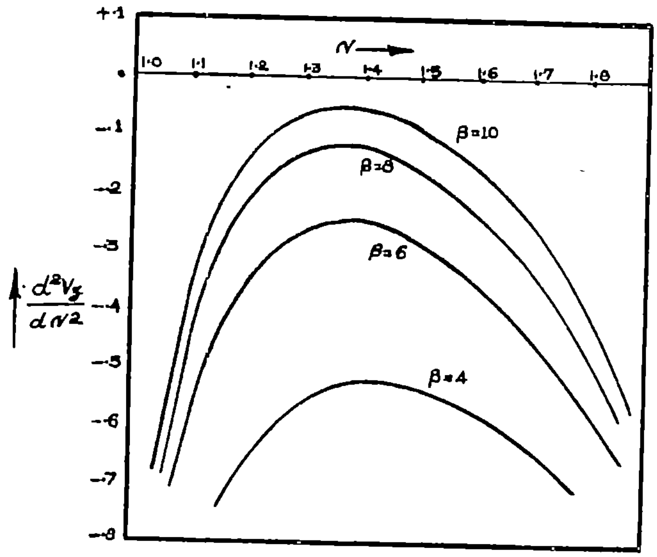

FIG. 1. IT SHOWS THAT FOR Θ = 2, POINT OF INFLEXION APPEARSONLY FOR β > 10

FIG. 2 IT SHOWS THE EXISTENCE OF POINTS OF INFLEXION FOR DIFFERENT VALUES OF β WHEN Θ = 5

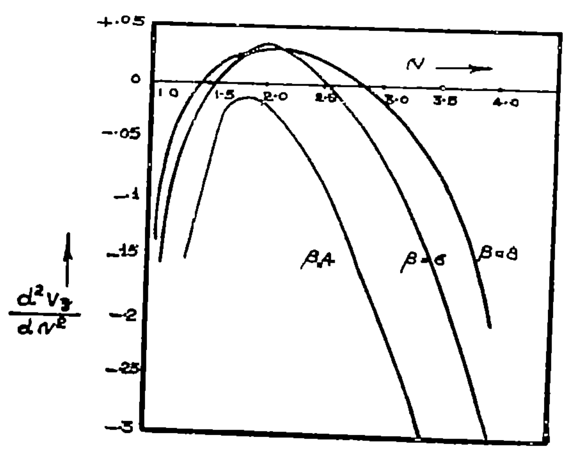

Fig. 1. shows that for $\theta = 2$, the point of inflexion appears only for values of $\beta > 10$. But as θ increases the limit of β after which point of inflexion appears first (β_c) goes on decreasing. It becomes clear when figures 1 and 2 are compared. When $\theta = 5$, β_c has been

shown (fig. 3) to be 4.16. For all values of $\beta > \beta_c$ there will essentially be two points of inflexion as the values of $\frac{d^2v_z}{d\gamma^2}$ for $\gamma = 1$ and $\gamma = \theta$ are always negative for all values of β and $\frac{d^2v_z}{d\gamma^2}$ vanishes as $\beta \to \infty$. The distance between the points of inflexion goes on increasing as β increases. But it appears that the two points of inflexion will always occur on one side of the point of maximum velocity.

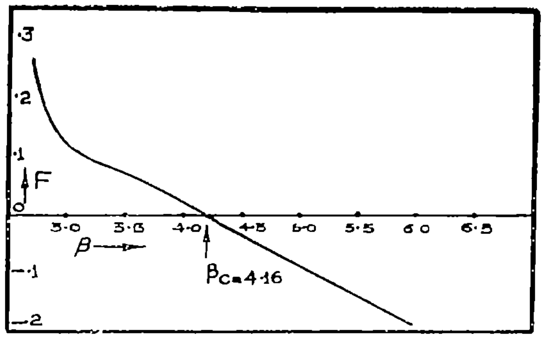

FIG. 3 IT GIVES β_C FOR $\theta = 5$

β_c for a given θ or θ_c for a given β can be obtained by the condition that the curve $\frac{d^2v_z}{d\gamma^2}$ touches the γ-axis. This will be given if the equations $\frac{d^2v_z}{d\gamma^2} = 0$ and $\frac{d^3v_z}{d\gamma^3} = 0$ have a common root. The later of these two equations gives the equation (18). If that is also a root of the equation $\frac{d^2v_z}{d\gamma^2} = 0$, the condition will be

$$F \equiv \left[\frac{\beta^2(\beta+1)}{\beta-2} \frac{\theta^\beta - \theta^2}{\theta^\beta - \theta^{-\beta}} \right]^{\frac{1}{\beta+2}} - \left[\frac{(\beta+1)(\beta+2)}{(\beta+1)(\beta-2)} \frac{\theta^\beta - \theta^2}{\theta^2 - \theta^{-\beta}} \right]^{\frac{1}{2\beta}} = 0 \tag{20}$$

which gives the critical value of the other when either of the two θ or β is specified.

The importance of the existence of the points of inflexion in the velcoity profile lies in the study of the problem of stability of the flow. Since $dv_z/d\gamma$ is non-zero except at the point of maximum velocity, therefore, there exist cylindrical vortex sheets within the flow, each element of which has the same vorticity, vorticity is decreasing from inner to the outer boundary. Appearance of the points of inflexion indicates the extremum of the vorticity. Though in the presence of dissipative terms, there is no theoretical result which shows the effect of a point of inflexion on the stability problem of the system. But there is an evidence of it (Berman (1958) in the experiments of the fluid flow through tubes with porouswalls. One more point is to be observed here. The induction current has only one component in θ—direction and vanishes on both the boundaries. The point of maximum velocity and maximum induction current coincide. This point can also be said as a point of inflexion of the magnetic field vector,

Asymmetry of the Velocity Profiles. In the absence of the magnetic field, the velocity profiles are given by

$$v_z = \frac{k}{4}\left[\frac{\theta^2-1}{\ln\theta}\ln\gamma-(\gamma^2-1)\right]. \tag{21}$$

Since the relation

$$v_z(\gamma, \theta) = v_z(1+\theta-\gamma, \theta) \tag{22}$$

is not independent of γ, therefore the velocity profile given by (21) is asymmetrical. Not only this, but if β is also present, from the equations (1) and (3), one observes that the relation

$$v_z(\gamma, \theta, \beta) = v_z\,(1+\theta-\gamma, \theta, \beta) \tag{23}$$

corresponding to the relation (22) is not independent of γ. Therefore again the velocity profiles are asymmetrical. We shall first study the asymmetry of the solution (21) and then see how this asymmetry differs from the asymmetry present in the solutions in the presence of magnetic field.

The point of maximum velocity γ_m of the solution (21) is given by

$$\gamma_m^2 = \frac{1}{2}\,\frac{\theta^2-1}{\ln\theta} \tag{24}$$

Table 1 gives the values of γ_m for different values of θ. It is clear from there that the point of maximum velocity always occurs nearer the inner boundary. Moreover as θ increases there is an actual shift of the point towards the boundary. This asymmetry is due to the geometrical asymmetry of the problem.

The asymmetry in the velocity profiles may be intuitively perceived by the following consideration. The radius of the inner cylinder being smaller than that of the outer one, the ratio of the viscous to the inertial forces is expected to be larger near the inner cylinder so that the velocity gradient will also be larger near the inner cylinder with the result that the maximum velocity will be reached in a shorter distance from the inner cylinder as compared with the distance of this point from the outer cylinder. By the same argument the larger the value of, we expect relatively closer to the inner cylinder will be the point of maximum velocity. If the geometrical asymmetry of the problem is removed that is if one considers limiting form of the solution (21) [Flow between two parallel plates], the velocity profile becomes symmetrical.

TABLE 1. It gives γ_m the point of maximum velocity for different values of $\theta(\beta=0)$ shift $d=\frac{\theta+1}{2}-\gamma_m$ from the middle of the annulus [It is positive if the shift is towards inner cylinder and negative, if the shift is towards outer cylinder.] and the percentage shift $\psi=\frac{d}{\frac{\theta-1}{2}}\times 100$. Every quantity has been evaluated upto three places of decimals.

θ	1.1	1.5	2	3	4	5	9
γ_m	1.050	1.242	1.471	1.909	2.325	2.731	4.268
d		.008	.029	.091	.175	.269	.732
ψ		3 2	5.8	9.1	11 66	13.45	18.3

One can easily show that the maximum velocity can never occur at the middle of the annulus. If so, θ is to be given by

$$f(\theta) \equiv e^{2\left(\frac{\theta-1}{\theta+1}\right)}-\theta = 0. \tag{25}$$

Consideration of the sign of the first and second derivative of $f(\theta)$ gives it to be continuously decreasing function as θ varies from one to infinity. Thus $\theta = 1$ is the only possible zero of $f(\theta)$ and so the point of maximum velocity can never be at the middle point of the annulus.

Similarly the maximum velocity occurs at the inner cylinder if and only if

$$F(\theta) \equiv \frac{1}{2}(\theta^2-1)-\ln\theta = 0 \tag{26}$$

whose first derivative $F'(\theta) \equiv (\theta^2-1)/\theta$ is positive for all values of θ. Therefore F is a continuously increasing function and $\theta = 1$ is the only possible zero. One concludes that in the absence of the magnetic field, the point of maximum velocity always lies between the inner cylinder and the middle point of the annulus.

This asymmetry in the velocity profile continues to be present even in the presence of magnetic field, but surprisingly the point of maximum velocity is shifted towards outer boundary. This point is given by

$$2\gamma^2(\theta^\beta-\theta^{-\beta})-\beta[(\theta^2-\theta^{-\beta})\,\gamma^\beta-(\theta^\beta-\theta^2)\gamma^{-\beta}] = 0 \tag{27}$$

when $\beta^2 \neq 4$ and

$$\frac{2\ln\gamma+1}{1+\gamma^{-4}} = \frac{2\theta^4\ln\theta}{\theta^4-1} \tag{28}$$

when $\beta^2 = 4$. Had there been no β, for a given value of θ, the point of maximum velocity would have occurred near the inner boundary. The magnetic field shifts the point towards outer boundary. The shift increases with the increase in β. Fig. 4 shows how the point of maximum velocity shifts towards outer cylinder with the increase in β. For $\theta = 5$, $\beta = 10$, the ratio of the distances to the point γ_m from outer and inner boundaries is approximately 3/13 while for $\theta = 5$, $\beta = 4$ this ratio is 5/11. We expect, naturally, that for given θ there

will be some value of β, for which the point of maximum velocity lies at the centre of the annulus. The values of such β's are given by putting $\gamma = (1+\theta)/2$ in the relation (27). We shall obtain

$$2^{\beta-1}(\theta^{2\beta}-1)(1+\theta)^{\beta+2}-\beta(\theta^{\beta+2}-1)(1+\theta)^{2\beta}+2^{2\beta}\beta(\theta^{2\beta}-\theta^{\beta+2}) = 0. \quad (29)$$

Taking the analogy for $\beta = 0$, we except that for a specified $\beta \neq 0$, the point of maximum velocity would move towards inner cylinder with the increase in θ. But, in fact, the result is just opposite; it still shifts towards outer cylinder. It becomes clear when figures 4 and 5 are compared. For $\theta = 4$, $\beta = 10$ the ratio of the distances to the point γ_m from outer and inner cylinders is approximately 1/4. Since this ratio for $\theta = 5$, $\beta = 10$ is 3/13, therefore the point of maximum velocity for $\theta = 5$ is nearer to the outer boundary than that for $\theta = 4$. The reason for it lies in the interaction of magnetic and viscous forces.

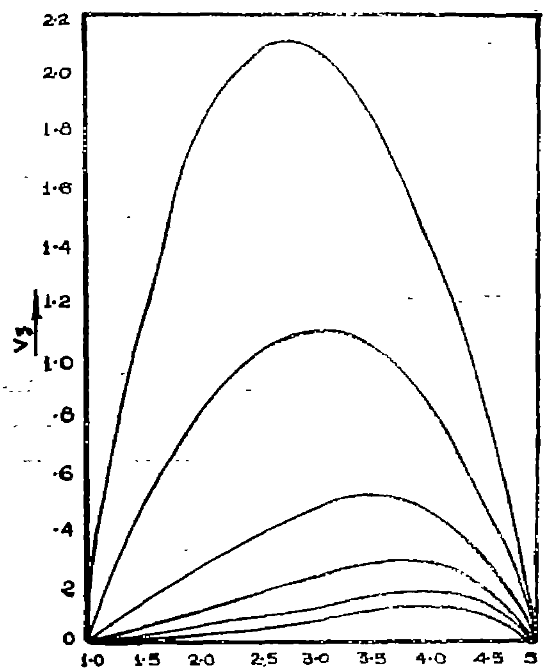

FIG. 4. VELOCITY PROFILES FOR θ = 5

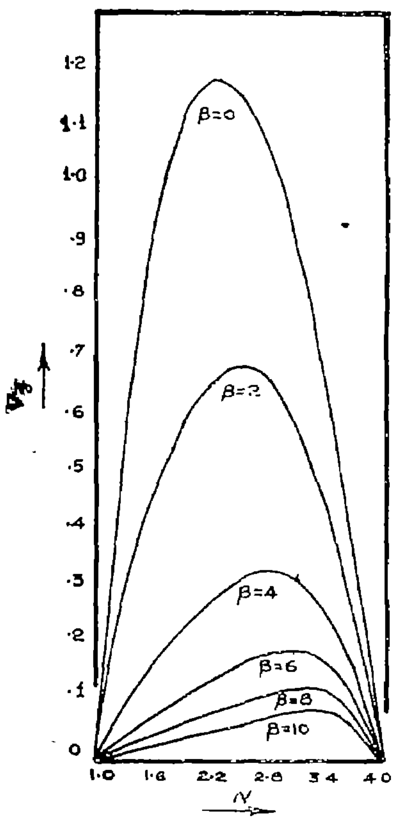

FIG. 5 VELOCITY PROFILES FOR θ = 5

The magnetic force is proportional to $\mu^2\sigma\mathcal{H}^2 V$ where $\mathcal{H}$ is the characteristic magnetic field (ω/r) perpendicular to the velocity field and the viscous force is proportional to $\rho\nu\, v/L^2$. The ratio of the magnetic force to the viscous force is given by Hartmann's number M

$$M = \frac{\beta\theta^2}{\gamma^2} = \frac{\mu^2\omega^2\sigma}{\rho\nu}\frac{\theta^2}{r^2} \quad (30)$$

where θ has been taken as the characteristic length of the flow. Magnetic forces and viscous forces, both are dissipative forces; velocity falls in their presence. But while magnetic force reduces the velocities considerably throughout the flow region, viscous force is more effective near the boundaries. The comparison of the values of M from (30) clearly indicates that it is greater near inner boundaries than outer one. Consequently, while the reduction of velocity near inner boundary is more due to induction drag, it is due to viscous drag near outer boundary. And we expect that the point of maximum velocity to occur near the outer cylinder. Expression (30) shows that for fixed ω/ν, the shift is again towards outer cylinder.

Effect on the Thickness of Boundary Layer. As has been seen already, the interaction between viscous drag and induction drag shifts the point of maximum velocity near the outer boundary. Moreover, since shift increases with β and θ, therefore for large values of β and θ, while in most of the region of the annulus velocity increases slowly, it falls down abruptly from maximum to zero in a narrow region near the outer boundary. From Fig. 4 one observes that for $\beta = 10$, $\theta = 5$, in less than one-fourth of the annulus the velocity falls from maximum to zero, while in more than three-fourths of the annulus it increases from zero to maximum. For $\beta = 4$, $\theta = 5$, the sharp decrease is in about 3/8 of the total region. Thus, though the velocity as a whole decreases markedly for large values of β, but simultaneously it provides a narrow region near outer boundary where the velocity gradient is sufficiently large. It can be argued that the thickness of the boundary layer near outer cylinder is increased markedly when large magnetic fields are present while the inner boundary layer is almost unaffected.

Since, if β is kept constant and θ allowed to vary, the point of asymmetry again shifts towards the outer cylinder. So we expect some further increase in the thickness of the boundary layer. In conclusion, when θ and β both are large, this region is more significant as could easily be seen by comparing the figures 4 and 5.

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ON MUKHOTI'S THEOREMS

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Generalizations of Rolle's theorem, the law of the mean, Cauchy's generalized law of the mean, and Darboux's mean value theorem for derivatives have been given by S. A. Mukhoti (1958). These generalizations are of particular interest since they do not require the existence of the derivative and, hence, may serve us when there is no derivative. The purpose of this paper is to strengthen and to extend Mukhoti's results. which are contained in the first parts of our first, second, fourth, and fifth theorems. Theorems of Karamata (1951) and Vučkovic′ (1952) as corrected by Utz (1955) are special cases of our results.

Let f be a function defined on a finite closed interval $a \leqslant x \leqslant b$. Throughout this paper we will use $\bar{D}_r f$, $\bar{D}_l f$, $\underline{D}_r f$, and $\underline{D}_l f$ to denote the upper right, upper left, lower right, and lower left Dini derivates [Karamata(1951), p. 188; McShane (1944), p. 87], respectively, of f. We use the name derivate to avoid confusion with one-sided derivatives [McShane (1944), p. 189; Vučkovic′ (1952), p. 159]. In addition, if p and q are real numbers, we will use

$$pD_r f+qD_l f$$

to denote any one of the expressions

$$p_1\bar{D}_r f+q_1\bar{D}_l f, \qquad p_2\bar{D}_r f+q_2\underline{D}_l f, \qquad p_3\underline{D}_r f+q_3\bar{D}_l f, \qquad p_4\underline{D}_r f+q_4\underline{D}_l f.$$

Also, we will use $p\bar{\underline{D}}_r f+q\bar{\underline{D}}_l f$ to denote all four of the above expressions. Finally, we will use braces to indicate alternate statements.

Our first theorem is a generalization of Rolle's theorem. Since we require only that $f(x_0) \geqslant f(a)$ $\{f(x_0) \leqslant f(a)\}$, the first part of the theorem is slightly stronger than Mukhoti's Theorem 1 [Mukhoti (1958), p. 87]. This enables us to strengthen later results.

Theorem 1. (*i*) *If f is upper {lower} semi-continuous at each point of the finite closed interval $a \leqslant x \leqslant b$, if $f(a) = f(b)$, and if there is an x_0, $a < x_0 < b$, such that $f(x_0) \geqslant f(a)$ $\{f(x_0) \leqslant f(a)\}$, then there exists a ξ, $a < \xi < b$ such that*

$$\bar{D}_r f(\xi) \leqslant 0 \leqslant \underline{D}_l f(\xi) \qquad \{\bar{D}_l f(\xi) \leqslant 0 \leqslant \underline{D}_r f(\xi)\}.$$

(*ii*) *If, in addition, f has a right derivate $D_r f$ and a left derivate $D_l f$ each of which is finite at ξ, then there are numbers p and q with $p \geqslant 0$, $q \geqslant 0$, $p+q = 1$ for which*

$$pD_r f(\xi)+qD_l f(\xi) = 0.$$

Proof. (*i*) A proof of this part may be obtained by making minor modifications in Mukhoti's proof [McShane (1944), p. 88]. The details will be left to the reader.

(*ii*) If $D_r f(\xi)$ and $D_l f(\xi)$ are both finite, we see that $D_r f(\xi)$. $D_l f(\xi) \leqslant 0$. If, in addition, $D_r f(\xi) = 0$, we may choose $p = 1$ and $q = 0$. If $D_r f(\xi) \neq 0$, we may choose

$$p = \frac{D_l f(\xi)}{D_l f(\xi)-D_r f(\xi)} \quad \text{and} \quad q = \frac{D_r f(\xi)}{D_r f(\xi)-D_l f(\xi)}.$$

In either case $pD_r f(\xi)+qD_l f(\xi) = 0$ which was to be shown.

Theorem 2 is a generalization of the law of the mean. Its proof, as well as the proofs of our remaining theorems, is based on Theorem 1.

Theorem 2. (*i*) *If f is upper {lower} semi-continuous at each point of the finite closed interval $a \leqslant x \leqslant b$ and if there is an x_0, $a < x_0 < b$, for which*

$$f(x_0) - \frac{f(b)-f(a)}{b-a}\, x_0 \geqslant \frac{bf(a)-af(b)}{b-a}$$

$$\left\{f(x_0) - \frac{f(b)-f(a)}{b-a}\, x_0 \leqslant \frac{bf(a)-af(b)}{b-a}\right\},$$

then there exists a ξ, $a < \xi < b$, such that

$$\underline{D}_l f(\xi) \geqslant \frac{f(b)-f(a)}{b-a} \geqslant \bar{D}_r f(\xi) \left\{\bar{D}_l f(\xi) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant \underline{D}_r f(\xi)\right\}.$$

(*ii*) *If, in addition, f has a right derivate $D_r f$ and a left derivate $D_l f$ each of which is finite at ξ, then there exist numbers p and q with $p \geqslant 0$, $q \geqslant 0$ $p+q = 1$ for which*

$$\frac{f(b)-f(a)}{b-a} = pD_r f(\xi)+qD_l f(\xi).$$

Proof. (*i*) A proof of this part may be obtained by making minor modifications in Mukhoti's proof [Mukhoti (1958), p. 89-90]. The details will be left to the reader.

(*ii*) If $D_r f(\xi)$ and $D_l f(\xi)$ are both finite, the auxilliary function

$$F(x) = f(x) - \frac{f(b)-f(a)}{b-a}\, x$$

satisfies the conditions of Theorem 1, part (ii). Hence, there are numbers p and q with $p \geqslant 0$, $q \geqslant 0$, $p+q = 1$ such that $pD_r F(\xi)+qD_l F(\xi) = 0$.
If follows that [McShane (1944), p. 191]

$$\frac{f(b)-f(a)}{b-a} = pD_r f(\xi)+qD_l f(\xi)$$

which was to be shown.

Theorem 3 is an extension of Taylor's formula with remainder.

Theorem 3. (*i*) *Let the function f, defined on the finite closed interval $a \leqslant x \leqslant b$, be such that its first $n-1$ derivatives exist and are upper {lower} semi-continuous on $a \leqslant x \leqslant b$ (only right and left derivatives are required at a and b, respectively) and let*

$$K = f(b)-\sum_{i=0}^{n-1} \frac{(b-a)^i f^{(i)}(a)}{i\,!} .$$

If v is a number such that $n-v > 0$ and if there is an x_0, $a < x_0 < b$, for which

$$\sum_{i=0}^{n-1} \frac{(b-x_0)^i f^{(i)}(x_0)}{i\,!} + \frac{(b-x_0)^{n-v}}{(b-a)^{n-v}} K \geqslant f(b)$$

$$\left\{ \sum_{i=0}^{n-1} \frac{(b-x_0)^i f^{(i)}(x_0)}{i\,!} + \frac{(b-x_0)^{n-v}}{(b-a)^{n-v}} K \leqslant f(b) \right\},$$

then there exists a ξ, $a < \xi < b$, such that

$$\underline{D}_l f^{(n-1)}(\xi) \geqslant \frac{(n-1)!(n-v)}{(b-\xi)^v(b-a)^{n-v}} K \geqslant \bar{D}_r f^{(n-1)}(\xi)$$

$$\left\{ \bar{D}_l f^{(n-1)}(\xi) \leqslant \frac{(n-1)!(n-v)}{(b-\xi)^v(b-a)^{n-v}} K \leqslant \underline{D}_r f^{(n-1)}(\xi). \right.$$

(*ii*) *If, in addition, $f^{(n-1)}$ has a right derivate $D_r f^{(n-1)}$ and a left derivate $D_l f^{(n-1)}$ each of which is finite at ξ, then there exist numbers p and q with $p \geqslant 0$, $q \geqslant 0$, $p+q = 1$ for which*

$$f(b) = f(a)+(b-a)f'(a)+\ldots+\frac{(b-a)^{n-1}f^{(n-1)}(a)}{(n-1)!}$$

$$+\frac{(b-a)^{n-v}(b-\xi)^v}{(n-v)(n-1)!} [pD_r f^{(n-1)}(\xi)+qD_l f^{(n-1)}(\xi)].$$

Proof. A proof of this theorem can be obtained by noting that the auxilliary function

$$F(x) = -f(b)+f(x)+(b-x)f'(x)+\dots$$
$$+\frac{(b-x)^{n-1}}{(n-1)!}\, f^{(n-1)}(x)+\frac{(b-x)^{n-\nu}}{(b-a)^{n-\nu}}\, K$$

satisfies the conditions of Theorem 1. The details of the proof are left to the reader.

Our next theorem is an extension of Darboux's intermediate value theorem for derivatives.

Theorem 4. (*i*) *If f is upper {lower} semi-continuous at each point of the finite closed interval $a \leqslant x \leqslant b$ and if k is a number such that*

$$\underline{D}_l f(b) < k < \bar{D}_r f(a) \quad \{\bar{D}_l f(b) > k > \underline{D}_r f(a)\},$$

then there exists a ξ, $a < \xi < b$, for which

$$D_r f(\xi) \leqslant k \leqslant D_l f(\xi) \qquad \{D_l f(\xi) \leqslant k \leqslant D_r f(\xi)\}.$$

(*ii*) *If, in addition, f has a right derivate $D_r f$ and a left derivate $D_l f$ each of which is finite at ξ, then there are numbers p and q with $p \geqslant 0$, $q \geqslant 0$, $p+q = 1$ for which*

$$pD_r f(\xi)+qD_l f(\xi) = k \qquad \{pD_r f(\xi)+qD_l f(\xi) = k\}.$$

Proof. (*i*) Let f be upper semi-continuous on $a \leqslant x \leqslant b$ and define the function F by $F(x) = f(x)-kx$. Then $\underline{D}_l F(b) < 0$ and $\bar{D}_r F(a) < 0$. Hence there is a point ξ, $a < \xi < b$, such that F has a maximum at ξ. Then $D_r F(\xi) \leqslant 0 \leqslant D_l F(\xi)$. It follows that $D_r f(\xi) \leqslant k \leqslant D_l f(\xi)$. A similar proof may be given when f is lower semi-continuous.

(*ii*) If $D_r f(\xi)$ and $D_l f(\xi)$ are finite, then $D_r F(\xi)$ and $D_l F(\xi)$ are finite and by Theorem 1, part (*ii*), there exist numbers $p \geqslant 0$, $q \geqslant 0$, $p + q = 1$ such that $pD_r F(\xi)+pD_l F(\xi) = 0$. Hence

$$pD_r f(\xi)+qD_l f(\xi) = k$$

which was to be shown. A similar proof may be given for f lower semi-continuous.

Let f and g be functions which are defined at each point of the closed interval $a \leqslant x \leqslant b$. We shall say that the derivate Df is partially bounded above by the derivate Dg provided there exists a number ξ, $a < \xi < b$, such that $[g(b)-g(a)]Df(\xi) \leqslant [f(b)-f(a)]Dg(\xi)$. In addition, we shall say that $D_r f+D_l f$ is partially bounded above by $D_r g+D_l g$ provided there exist numbers p, q, and ξ with $p \geqslant 0$, $q \geqslant 0$, $p+q = 1$, and $a < \xi < b$ such that

$$[g(b)-g(a)][pD_r f(\xi)+qD_l f(\xi)] \leqslant [f(b)-f(a)][pD_r g(\xi)+qD_l g(\xi)].$$

Moreover, we shall say that the four expressions $\bar{\underline{D}}_r f+\bar{\underline{D}}_l f$ are partially bounded above by $D_r g+D_l g$ provided each of the expressions is partially bounded above by $D_r g+D_l g$. The notion of partially bounded below is defined by replacing "above" by "below" and "$\leqslant$" by "$\geqslant$" in the preceding definitions.

Our final results, which are extensions of Cauchy's generalized law of the mean. will be stated as three related theorems. We shall prove only one relation from the first theorem. The other relations can be similarly proved. The reader will observe that additional relations may be obtained by interchanging upper and lower and reversing the inequalities in the theorems.

Theorem 5a. (*i*) *If the function f is upper semi-continuous and the function g is lower semi-continuous on the closed interval $a \leqslant x \leqslant b$, if $f(b)-f(a) \geqslant 0$, if $g(b)-g(a) \geqslant 0$, and if there is an x_0, $a < x_0 < b$, for which*

$$[g(b)-g(a)][f(x_0)-f(a)] \geqslant [f(b)-f(a)][g(x_0)-g(a)]$$

then $\underline{D}_r f$ and $\bar{D}_r f$ are partially bounded above by $D_r g$ and $\bar{D}_r g$, respectively, and $\bar{D}_l f$ and $\underline{D}_l f$ are partially bounded below by $D_l g$ and $\underline{D}_l g$, respectively

(*ii*) *If, in addition, the one sided upper and lower derivates of f and g are finite at ξ, then $\bar{\underline{D}}_r f+\bar{\underline{D}}_l f$ is partially bounded above by $\bar{D}_r g+\bar{D}_l g$ and is partially bounded below by $\underline{D}_r g$ $+\underline{D}_l g$. Moreover, $\bar{D}_r f+\underline{D}_l f$ and $\underline{D}_r f+\bar{D}_l f$ are partially bounded above and below by $\bar{D}_r g+$ $\underline{D}_l g$ and $\underline{D}_r g+\bar{D}_l g$, respectively.*

Proof. (*i*) Let

$$F(x) = [g(b)-g(a)][f(x)-f(a)]-[f(b)-f(a)][g(x)-g(a)].$$

Since F satisfies the conditions of Theorem 1, part (*i*), there exists a ξ, $a < \xi < b$, such that $\bar{D}_r F(\xi) \leqslant 0 \leqslant \underline{D}_l F(\xi)$. It follows that the relations of part (*i*) hold.

(*ii*) If all derivatives of f and g are finite at ξ, then the function F satisfies the conditions of Theorem 1, part (*ii*). Hence there exist numbers p and q with $p \geqslant 0$, $q \geqslant 0$, $p+q = 1$ such that $p\bar{D}_r F(\xi)+q\underline{D}_l F(\xi) = 0$. But [McShane (1944), p. 190]

$$\bar{D}_r F(x) \geqslant [g(b)-g(a)]\bar{D}_r f(x)-[f(b)-f(a)]\bar{D}_r g(x)$$

and

$$\underline{D}_l F(x) \geqslant [g(b)-g(a)]\underline{D}_l f(x)-[f(b)-f(a)]\bar{D}_l g(x).$$

It follows that

$$[g(b)-g(a)][p\bar{D}_r f(\xi)+q\underline{D}_l f(\xi)] \leqslant [f(b)-f(a)][p\bar{D}_l g(\xi)+q\bar{D}_l g(\xi)]$$

which was to be shown.

Theorem 5b. (*i*) *If the function f is lower semi-continuous and the function g is upper semi-continuous on the closed interval $a \leqslant x \leqslant b$ if $f(b)-f(a) \leqslant 0$, if $g(b)-g(a) \leqslant 0$, and if there is an x_0, $a < x_0 < b$, for which*

$$[g(b)-g(a)][f(x_0)-f(a)] \geqslant [f(b)-f(a)][g(x_0)-g(a)],$$

then $\underline{D}_r f$ and $\bar{D}_r f$ are partially bounded above by $\underline{D}_r g$ and $D_r g$, respectively, and $\bar{D}_l f$ and $\underline{D}_l f$ are partially bounded below by $\bar{D}_l g$ and $D_l g$ respectively.

(*ii*) *If, in addition, the one-sided upper and lower derivates of f and g are finite at ξ, then $\bar{D}_r f+\bar{D}_l f$ is partially bounded above by $\underline{D}_r g+\underline{D}_l g$ and is partially bounded below by $\bar{D}_r g+\bar{D}_l g$. Moreover, $\bar{D}_r f+\underline{D}_l f$ and $\underline{D}_r f+\bar{D}_l f$ are partially bounded above and below by $\bar{D}_r g+\underline{D}_l g$ and $\underline{D}_r g+\bar{D}_l g$, respectively.*

Theorem 5c. (*i*) *If the functions f and g are upper semi-continuous on the finite closed interval $a \leqslant x \leqslant b$, if $f(b)-f(a) \leqslant 0$, if $g(b)-g(a) \geqslant 0$, and if there is an x_0 $a < x_0 < b$, for which*

$$[g(b)-g(a)][f(x_0)-f(a)] \geqslant [f(b)-f(a)][g(x_0)-g(a)],$$

then $\bar{D}_r f$ and $\underline{D}_r f$ are partially bounded above by $\underline{D}_r g$ and $D_r g$, respectively, and $\underline{D}_l f$ and $\bar{D}_l f$ are partially bounded below by $\bar{D}_l g$ and $D_l g$, respectively.

(*ii*) *If, in addition, the one-sided upper and lower derivates of f and g are finite at ξ, then $\bar{D}_r f+\underline{\bar{D}}_l f$ and $\bar{D}_r g+\underline{\bar{D}}_l g$ are partially bounded above by $D_r g+\underline{D}_l g$ and $\bar{D}_r f+\bar{D}_l f$, respectively, and are partially bounded below by $\underline{D}_r g+\bar{D}_l g$ and $\underline{D}_r f+\underline{D}_l f$, respectively. Moreover $\bar{D}_r f+\underline{D}_l f$ and $\underline{D}_r f+\bar{D}_l f$ are partially bounded above and below by $\underline{D}_r g+\bar{D}_l g$ and $\bar{D}_r g+\underline{D}_l g$, respectively.*

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